Fracture mechanics

ME 524

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Disclaimer

 The content of this presentation are produced by R. Abedi and also taken from several sources, particularly from the fracture mechanics presentation by Dr. Vinh Phu Nguyen <u>nvinhphu@gmail.com</u> Monash University (formerly at University of Adelaide and Ton Duc Thang University)

Other sources

Books:

- T. L. Anderson, Fracture Mechanics: Fundamentals and Applications, 3rd Edition, CRC Press, USA, 2004
- S. Murakami, Continuum Damage Mechanics, Springer Netherlands, Dordrecht, 2012.
- L.B. Freund, Dynamic Fracture Mechanics, Cambridge University Press, 1998.
- B. Lawn, Fracture of Brittle Solids, Cambridge University Press, 1993.
- M.F. Kanninen and C.H. Popelar, Advanced Fracture Mechanics, Oxford Press, 1985.
- R.W. Hertzberg, Deformation & Fracture Mechanics of Engineering Materials. John Wiley & Sons, 2012.

Course notes:

- V.E. Saouma, Fracture Mechanics lecture notes, University of Colorado, Boulder.
- P.J.G. Schreurs, Fracture Mechanics lecture notes, Eindhoven University of Technology (2012).
- A.T. Zender, Fracture Mechanics lecture notes, Cornell University.
- L. Zhigilei, http://people.virginia.edu/~lz2n/mse209/index.html
 MSE 2090: Introduction to Materials Science Chapter 8, Failure

Outline

- Brief recall on mechanics of materials
 stress/strain curves of metals, concrete
- Introduction
- Linear Elastic Fracture Mechanics (LEFM)
 Energy approach (Griffith, 1921, Orowan and Irwin 1948)
 - Stress intensity factors (Irwin, 1960s)
- LEFM with small crack tip plasticity
 - Irwin's model (1960) and strip yield model
 - Plastic zone size and shape
- Elastic-Plastic Fracture Mechanics
 Crack tip opening displacement (CTOD), Wells 1963
 - J-integral (Rice, 1958)
- Dynamic Fracture Mechanics



Source: Sheiba, Olson, UT Austin





Outline (cont.)

Fatigue

- Fatigue crack propagation & life prediction
- Paris law



- **Ductile versus Brittle Fracture**
- Stochastic fracture mechanics
- Microcracking and crack branching in brittle fracture



Ductile Fracture





Stress intesity factor range, ΔK , (MPa \sqrt{m}), log scale



DynamicFractureBifurcation PMMA Murphy 2006



quippy documentation (www.jrkermode.co.uk)

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Outline (cont.)

Singular Element

- **Computational fracture mechanics**
- FEM aspects:
 - Isoparametric singular elements
 - Calculation of LEFM/EPFM Integrals
 - Adaptive meshing, XFEM
- Cohesive crack model (Hillerborg, 1976)
- Continuum Damage Mechanics
 - size effect (Bazant)

Cracks in FEM

P. Clarke UTSI







XFEM



http://www.fgg.uni-lj.si/~/pmoze/ESDEP/master/toc.htm

17

18

14

х





Adiabatically CRACK reated **Cohesive Zone** Decohesion Fracture Crack-tip craze fibril P. Leevers Imperical College COLD BULK POLYMER

5

underwood.faculty.asu.edu

Introduction







Cracks: ubiquitous !!!

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Definitions

- Crack, Crack growth/propagation
- A **fracture** is the (local) separation of an object or material into two, or more, pieces under the action of stress.
- Fracture mechanics is the field of <u>mechanics</u> concerned with the study of the propagation of cracks in materials. It uses methods of analytical <u>solid mechanics</u> to calculate the driving force on a crack and those of experimental solid mechanics to characterize the material's resistance to <u>fracture</u> (Wiki).

Objectives of FM



• What is the residual strength as a function of crack size?

- What is the critical crack size?
- How long does it take for a crack to grow from a certain initial size to the critical size?

Approaches to fracture

- Stress analysis
- Energy methods

covered in the course

- Computational fracture mechanics
- Micromechanisms of fracture (eg. atomic level)
- Experiments
- Applications of Fracture Mechanics

Design philosophies

• Safe life

The component is considered to be free of defects after fabrication and is designed to remain defect-free during service and withstand the maximum static or dynamic working stresses for a certain period of time. If flaws, cracks, or similar damages are visited during service, the component should be discarded immediately.

Damage tolerance

The component is designed to withstand the maximum static or dynamic working stresses for a certain period of time even in presence of flaws, cracks, or similar damages of certain geometry and size.



- FM plays a vital role in the design of every critical structural or machine component in which durability and reliability are important issues (aircraft components, nuclear pressure vessels, microelectronic devices).

- has also become a valuable tool for material scientists and engineers to guide their efforts in developing materials with improved mechanical properties.

Preliminaries History

Indicial notation

a 3D vector $\mathbf{x} = \{x_1, x_2, x_3\}$ $||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ two times repeated index=sum, summation/dummy index $\sigma_{xx}\epsilon_{xx} + \sigma_{xy}\epsilon_{xy} + \sigma_{yx}\epsilon_{yx} + \sigma_{yy}\epsilon_{yy}$

$$i = 1, 2, 3$$

$$||\mathbf{x}|| = \sqrt{x_i x_i}$$

$$||\mathbf{x}|| = \sqrt{x_k x_k}$$

$$\sigma_{ij}\epsilon_{ij} \ \sigma_{ij,j} = 0$$

 $\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial u} = 0, \quad \frac{\partial \sigma_y}{\partial u} + \frac{\partial \tau_{xy}}{\partial x} = 0$ i: free index (appears precisely once in $\sigma_{xx}n_x + \sigma_{xy}n_y = t_x$ each side of an equation) $\sigma_{ij}n_j = t_i$ $\sigma_{yx}n_x + \sigma_{yy}n_y = t_y$ tensor notation $\boldsymbol{\sigma}:\boldsymbol{\epsilon}$

Engineering/matrix notation

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad ||\mathbf{x}|| = \mathbf{x}^{\mathrm{T}} \mathbf{x} \qquad ||\mathbf{x}|| = \sqrt{x_i x_i}$$
Voigt notation
$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{bmatrix} \qquad \boldsymbol{\sigma}^{\mathrm{T}} \boldsymbol{\epsilon} = \sigma_{ij} \epsilon_{ij}$$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \qquad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{xy} & \epsilon_{yy} \end{bmatrix}$$

Stress/strain curve



Wikipedia

<u>necking</u>=decrease of cross-sectional area due to plastic deformation

1: ultimate tensile strength 17

Principal stresses



Principal stresses are those stresses that act on principal surface. Principal surface here means the surface where components of shear-stress is zero.

$$\sigma_1, \sigma_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

Material classification / Tensile test



Strain energy density $u = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \underbrace{\nu}_E^{\text{Poisson's ratio}}_E \sigma_x \sigma_y + \sigma_y \sigma_z$ $+ \sigma_z \sigma_x) + \frac{1}{2\mu}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)$

Plane problems $\mu = rac{E}{2(1+
u)}$ shear modulus

$$u = \frac{1}{4\mu} \left[\frac{\kappa + 1}{4} (\sigma_x^2 + \sigma_y^2) - 2(\sigma_x \sigma_y - \tau_{xy}^2) \right]$$

Kolosov coefficient

 $\kappa = \begin{cases} 3 - 4\nu & \text{plane strain} \\ \frac{3 - \nu}{1 + \nu} & \text{plane stress} \end{cases}$

3.1 Fracture modes3.2 Ductile fracture

Fracture

Fracture: separation of a body into pieces due to stress, at temperatures below the melting point.

Steps in fracture:

- crack formation
- crack propagation

Depending on the ability of material to undergo plastic deformation before the fracture two fracture modes can be defined - **ductile or brittle**

Lecture source:

Prof. Leonid Zhigilei, <u>http://people.virginia.edu/~lz2n/mse209/index.html</u> MSE 2090: Introduction to Mat²/erials Science Chapter 8, Failure

- **Ductile fracture** most metals (not too cold):
 - Extensive plastic deformation ahead of crack
 - Crack is "stable": resists further extension unless applied stress is increased
- Brittle fracture ceramics, ice, cold metals:
 - Relatively little plastic deformation
 - Crack is "unstable": propagates rapidly without increase in applied stress

Ductile fracture is preferred in most applications



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Brittle vs. Ductile Fracture

- **Ductile materials** extensive plastic deformation and energy absorption ("toughness") before fracture
- **Brittle materials** little plastic deformation and low energy absorption before fracture



Lecture source:

Prof. Leonid Zhigilei, <u>http://people.virginia.edu/~lz2n/mse209/index.html</u> MSE 2090: Introduction to Matérials Science Chapter 8, Failure

Brittle vs. Ductile Fracture



- A. Very ductile, soft metals (e.g. Pb, Au) at room temperature, other metals, polymers, glasses at high temperature.
- **B.** Moderately ductile fracture, typical for ductile metals
- C. Brittle fracture, cold metals, ceramics.

Lecture source:

Prof. Leonid Zhigilei, <u>http://people.virginia.edu/~lz2n/mse209/index.html</u> MSE 2090: Introduction to Materials Science Chapter 8, Failure

Fracture Types

Shearing

- Applied stress =>
- Dislocation generation and motion =>
- Dislocations coalesce at grain boundaries =>
- Forming voids =>
- Voids grow to form macroscopic cracks
- Macroscropic crack growth lead to fracture
 Plastic deformation (ductile material)







Dough-like or conical features





Ductile Fracture (Dislocation Mediated)



Lecture source:

Prof. Leonid Zhigilei, <u>http://people.virginia.edu/~lz2n/mse209/index.html</u> MSE 2090: Introduction to Mat^erials Science Chapter 8, Failure

Ductile Fracture



(Cup-and-cone fracture in Al)



Scanning Electron Microscopy: *Fractographic* studies at high resolution. Spherical "dimples" correspond to microvoids that initiate crack formation.

Lecture source:

Prof. Leonid Zhigilei, <u>http://people.virginia.edu/~lz2n/mse209/index.html</u> MSE 2090: Introduction to Materials Science Chapter 8, Failure

Brittle Fracture (Limited Dislocation Mobility)

- ➢ No appreciable plastic deformation
- Crack propagation is very fast
- Crack propagates nearly perpendicular to the direction of the applied stress
- Crack often propagates by cleavage breaking of atomic bonds along specific crystallographic planes (cleavage planes).





Brittle fracture in a mild steel

Lecture source:

Prof. Leonid Zhigilei, <u>http://people.virginia.edu/~lz2n/mse209/index.html</u> MSE 2090: Introduction to Materials Science Chapter 8, Failure

Brittle Fracture

- **A. Transgranular fracture**: Fracture cracks pass through grains. Fracture surface have faceted texture because of different orientation of cleavage planes in grains.
- **B. Intergranular fracture**: Fracture crack propagation is along grain boundaries (grain boundaries are weakened or embrittled by impurities segregation etc.)



Lecture source:

Prof. Leonid Zhigilei, <u>http://people.virginia.edu/~lz2n/mse209/index.html</u> MSE 2090: Introduction to Mat²erials Science Chapter 8, Failure

Fracture Types

Cleavage

mostly brittle





Fatigue



intra-granulair (or transgranular) split atom bonds inter-granulair between grain boundaries

 Cracks grow a very short distance every time
 Clam shell structures mark the location of crack tip after each individual cyclic loading



Fracture Types

Delamination (De-adhesion)



Crazing

- Common for polymers
- sub-micormeter voids initiate

stress whitening because of light reflection from crazes





3.3 Ductile to brittle transition

Ductile-to-brittle transition



Low temperatures can severely embrittle steels. The Liberty ships, produced in great numbers during the WWII were the first all-welded ships. A significant number of ships failed by catastrophic fracture. Fatigue cracks nucleated at the corners of square hatches and propagated rapidly by brittle fracture.

Lecture source:

Prof. Leonid Zhigilei, <u>http://people.virginia.edu/~lz2n/mse209/index.html</u> MSE 2090: Introduction to Materials Science Chapter 8, Failure

Charpy v-notch test



1. Temperature Effects

Temperature decrease => Ductile material can become brittle

- BCC metals: Limited dislocation slip systems at low T =>
- Impact energy drops suddenly over a relatively narrow temperature range around DBTT.
 - Ductile to brittle transition temperature (DBTT) or
 - Nil ductility transition temperature (T₀)
- FCC and HCP metals remain ductile down to very low temperatures
- Ceramics, the transition occurs at much higher temperatures than for metals



strain

Titanic in the icy water of Atlantic (BCC)
-39teel structures are every likely to fail in winter

Fracture surfaces of tested specimens



Source: Tapany Udomphol, Suranaree University of Technology

http://eng.sut.ac.th/metal/images/stories/pdf/14_Brittle_fracture_and_impact_testing_1-6.pdf

2. Impurities and alloying effect on DBTT

- Alloying usually increases DBTT by inhibiting dislocation motion. They are generally added to increase strength or are (an unwanted) outcome of the processing
- For steel P, S, Si, Mo, O increase DBTT while Ni, Mg decease it.



Decrease of DBTT by Mg: formation of manganesesulfide (MnS) and consumption of some S. It has some side effects
3. Radiation embrittlement through DBTT

- Energetic particles (such as neutron or fission fragments) => knocking atoms out of natural lattice positions changing material property
- T_0 corresponds to $K_{JC} = 100 \text{ MPa}\sqrt{m}$

400

Stress (Pa)

0

0

 Wallin Master Curve model: K-T shifts to right by ΔT_o by irradiation

Irradiated

Δ

0

Unirradiated

Yield stress

Elastic limit

Strain (%)

Ultimate tensile stress

1.5



Irradiation effect:

- 1. Strengthening
- 2. More brittle

3. Radiation embrittlement through DBTT



 $K_{Jc(med)} = 30 + 70 \exp[0.019(T - T_0)]$, MPa \sqrt{m}

4. Hydrogen embrittlement through DBTT

- Hydrogen in alloys drastically reduces ductility in most important alloys:
 - nickel-based alloys and, of course, both ferritic and austenitic steel
 - Steel with an ultimate tensile strength of less than 1000 Mpa is almost insensitive
- A very common mechanism in Environmentally assisted cracking (EAC):
 - High strength steel, aluminum, & titanium alloys in aqueous solutions is usually driven by hydrogen production at the crack tip (i.e., the cathodic reaction)
 - Different from previously thought anodic stress corrosion cracking(SCC)
- Reason (most accepted)
 - Reduces the bond strength between metal atoms => easier fracture.





Grains



Grain boundary barrier to dislocation motion: High angle grain boundaries block slip and harden the material







5. Grain size

- In BCC metals, brittle fracture can be initiated by dislocation glide within a crystalline grain
- Yield stress depends on grain size (Hall-Petch law)

$$\sigma_y = \sigma_0 + \frac{k_y}{\sqrt{d}}$$

- Dislocation pile-up acts as crack with size ≈d =>
- Stress to cause brittle fracture is

$$\sigma_f = \frac{k_f}{\sqrt{d}}, \ k_f = \sqrt{\frac{EG_c}{\pi}}$$

- Left size: small d (d < d_{cr}) $\sigma_y < \sigma_f \Rightarrow$ Ductile fracture
- Right side: large (d > d_{cr}) $\sigma_y < \sigma_f \Rightarrow$ Britle fracture

- Small grain size =>
 - 1. Lower DBTT (more ductile)
 - **2.** Increases thoughness

⇒ DUCTILITY & STRENGH INCREASE SIMULTANEOUSLY!!!! ONLY STRENGTHING MECHANISM THAT IMPROVES DUCTILITY



 σ

Lowering Grain size

- Small grain size =>
 - **1. Lower DBTT (more ductile)**
 - **2.** Increases thoughness

⇒

DUCTILITY & STRENGH INCREASE SIMULTANEOUSLY!!!! ONLY STRENGTHING MECHANISM THAT IMPROVES DUCTILITY

 Grain boundaries have higher energies (surface energy) ⇒ Grains tend to diffuse and get larger to lower the energy
Heat treatments that provide grain refinement such as air cooling, recrystallisation during hot working help to lower transition temperature.

cr

DBTT relation to grain size analysis



6.Size effect and embrittlement

- Experiment tests: scaled versions of real structures
- The result, however, depends on the size of the specimen that was tested
- From experiment result to engineering design: knowledge of size effect required
- The size effect is defined by comparing the nominal strength (nominal stress at failure) σ_N of geometrically similar structures of different sizes.
- Classical theories (elastic analysis with allowable stress): cannot take size effect into account
- LEFM: strong size effect

- Size effect is crucial in concrete structures (dam, bridges), geomechanics (tunnels): laboratory tests are small
- Size effect is less pronounced in mechanical and aerospace engineering the structures or structural components can usually be tested at full size.

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geometrically similar structures of different sizes



b is thickness



Fig. 1. Specimens with sizes in a scale range of 1:32 and specimen proportions.

| type | Α | В | С | D | Е | F | |
|--------|-------|------|-----|-----|-----|------|--|
| D [mm] | 50 | 100 | 200 | 400 | 800 | 1600 | |
| r [mm] | 36.25 | 72.5 | 145 | 290 | 580 | 1160 | |

Structures and tests



Usual lab tests (10 cm)



Unusual lab tests (1m)



"Usual" structures (10m)



Size effect (cont.)



 Large structures are softer than small structures.
A large structure is more brittle and has a lower strength than a small structure.

Size effect



For very small structures the curve approaches the horizontal line and, therefore, the failure of these structures can be predicted by a strength theory. On the other hand, for large structures the curve approaches the inclined line and, therefore, the failure of these structures can be predicted by LEFM.

Bazant's size effect law

$$(\sigma_N)_u = Af_t \left(1 + \frac{W}{B}\right)^{-1/2}$$
(14.8)

where

- $(\sigma_N)_u$ = Nominal stress at failure of a structure of specific shape and loading condition.
 - W = Characteristic length of the structure.
- A, B = Positive constants that depend on the fracture properties of the material and on the shape of the structure, but not on the size of the structure.
 - f_t = Tensile strength of the material introduced for dimensional purposes.

7. Rate effects on ductility

- Same materials that show temperature toughness sensitivity (BCC metals) show high rate effect
- Polymers are highly sensitive to strain rate (especially for T > glass transition temperature)



Strain rate effects on Impact toughness

Strain rate **オ** ⇒ DBTT **オ** (more brittle in impact)



FIGURE 4.5 Effect of loading rate on the cleavage fracture toughness of a structural steel. Taken from Barsom, J.M., "Development of the AASHTO Fracture Toughness Requirements for Bridge Steels." *Engineering Fracture Mechanics*, Vol. 7, 1975, pp. 605–618.

Strain rate effects on crack resistance



- Strain rate 🐬
- J **7** (upper shelf of toughness; opposite to impact toughness):
 - Ductile fracture of metals is primarily strain controlled.
 - J integral is elevated by high strain rates

Crack speed effect on dynamic crack propagation resistance



- Strain speed **7**
- K_{ID} 7 (Insensitive at low speeds, quick increase approaching V_I)
- Increasing toughness makes K_{ID} more sensitive and grow faster

$$K_{ID} = \frac{K_{IA}}{1 - \left(\frac{V}{V_l}\right)^m}$$

8. Triaxial stress and confinement

Effect of specimen thickness

• Larger specimen size (in-service components) provides higher constraint → more brittle.



Source: Tapany Udomphol, Suranaree University of Technology

http://eng.sut.ac.th/metal/images/stories/pdf/14 Brittle fracture and impact testing 1-6.pdf

May-Aug 2007

Ductile to brittle transition

Often hardening (increasing strength) reduces ductility Phenomena affecting ductile/brittle response

- 1. T (especially for BCC metals and ceramics)
- 2. Impurities and alloying
- 3. Radiation
- 4. Hydrogen embrittlement
- 5. Grain size
- 6. Size effect
- 7. Rate effect
- 8. Confinement and triaxial stress state

Decreasing grain size is the only mechanism that hardens and promotes toughness

4. Linear Elastic Fracture Mechanics (LEFM)

4.1Griffith energy approach

Atomistic view of fracture



$$\mathbf{P} = -\frac{\partial \Pi_a}{\partial \mathbf{r}} \quad \Rightarrow \ P = \frac{\partial \Pi_a}{\partial r} \ (1\mathrm{D})$$

- P: Force between atoms (tensile positive cancels sign)
- Position r: Distance from other atom
- x_0 : Equilibrium position, $P = \frac{\partial \Pi_a}{\partial r} = 0$
- Displacement $x = r x_0$.
- λ : Length scale where atomistic force is too small.

•
$$P_c$$
: Max force at $\frac{\partial^2 \Pi_a}{\partial r^2} = 0$.



$$\begin{cases} \text{Continuum:} \\ E \\ \gamma_c(\text{surface energy}) \end{cases}$$

$$\Rightarrow \begin{array}{ll} \sigma_c &=& \frac{nP_c}{A} \\ \sigma &=& \sigma_c \sin(\frac{\pi \epsilon x_0}{\lambda}) \end{array}$$

Atomistic view of fracture



- Compare $\gamma_s = \frac{W_s}{A}$ with $\sigma = \frac{\Sigma P_i}{A}$
- Factor $\frac{1}{2}$: fracture generates two surfaces.

Stress concentration



Geometry discontinuities: holes, corners, notches, cracks etc: stress concentrators/risers

Stress concentration (cont.)



Elliptic hole

Inglis, 1913, theory of elasticity





 $b = \frac{b^2}{a} \qquad \sigma_A = \sigma \left(1 + 2\sqrt{\frac{a}{\rho}}\right)$ $1 \le \infty$ Stress concentration factor [-]

radius of curvature

$$K = \frac{\sigma_A}{\sigma} = 1 + \frac{2a}{b}$$

Griffith's work (brittle materials)

FM was developed during WWI by English aeronautical engineer A. A. Griffith to explain the following observations:

- The stress needed to fracture bulk <u>glass</u> is around 100 MPa
- The theoretical stress needed for breaking atomic bonds is approximately 10,000 MPa
- experiments on glass fibers that Griffith himself conducted: the fracture stress increases as the fiber diameter decreases
 => Hence the uniaxial tensile strength, which had been used extensively to predict material failure before Griffith, could not be a specimen-independent material property.

Griffith suggested that the low fracture strength observed in xperiments, as well as the size-dependence of strength, was due to the presence of **microscopic flaws** in the bulk material.



Griffith's size effect experiment

| Diameter (10 ⁻³ in) | Breaking stress (lb/in ²) | Diameter (10 ⁻³ in) | Breaking stress (lb/in ²) | |
|-----------------------------------|--|-----------------------------------|--|--|
| 40.00 | 24 900 | 0.95 | 117 000 | |
| 4.20 | 42 300 | 0.75 | 134 000 | |
| 2.78 | 50 800 | 0.70 | 164 000 | |
| 2.25 | 64 100 | 0.60 | 185 000 | |
| 2.00 | 79 600 | 0.56 | 154 000 | |
| 1.85 | 88 500 | 0.50 | 195 000 | |
| 1.75 | 82 600 | 0.38 | 232 000 | |
| 1.40 | 85 200 | 0.26 | 332 000 | |
| 1.32 | 99 500 | 0.165 | 498 000 | |
| 1.15 | 88 700 | 0.130 | 491 000 | |

TABLE 1.1. Strength of glass fibers according to Griffith's experiments.

"the weakness of isotropic solids... is due to the presence of discontinuities or flaws... The effective strength of technical materials could be increased 10 or 20 times at least if these flaws could be eliminated."

Fracture stress: discrepancy between theory and experiment

| $\sigma_{th} = \sqrt{\frac{E\gamma}{a_0}}$ | a ₀ [m] | E [GPa] | σ _{th} [GPa] | σ _b [MPa] | σ_{th}/σ_b |
|--|-----------------------|---------|-----------------------|----------------------|------------------------|
| glass | 3 * 10 ⁻¹⁰ | 60 | 14 | 170 | 82 |
| stee | 10 ⁻¹⁰ | 210 | 45 | 250 | 180 |
| silica fibers | 10 ⁻¹⁰ | 100 | 31 | 25000 | 1.3 |
| iron whiskers | 10 ⁻¹⁰ | 295 | 54 | 13000 | 4.2 |
| silicon whiskers | 10 ⁻¹⁰ | 165 | 41 | 6500 | 6.3 |
| alumina whiskers | 10 ⁻¹⁰ | 495 | 70 | 15000 | 4.7 |
| ausformed steel | 10 ⁻¹⁰ | 200 | 45 | 3000 | 15 |
| piano wire | 10 ⁻¹⁰ | 200 | 45 | 2750 | 16.4 |

Cause of discrepancy: 1. Stress approach



 \Leftrightarrow

• For a large domain $(L, W \ll a)$, crack length 2a, & radius of curvature $\rho \ll a$:

$$\sigma_m = \sigma \left(1 + 2\sqrt{\frac{a}{\rho}} \right) \approx 2\sigma \sqrt{\frac{a}{\rho}}$$

• For effective remote stress at failure $\sigma = \sigma_f$, σ_m is equal to atomistic based σ_c :

$$\sigma_c = 2\sigma_f \sqrt{\frac{a}{\rho}} \qquad \sigma_f = \sqrt{\frac{E\gamma_s}{4a}\frac{\rho}{x_0}}$$

• If we assume the crack is sharp at atomistic level $\rho \approx x_0$ we get,

$$\sigma_f = \sqrt{\frac{E\gamma_s}{4a}}$$

Atomistic:

 $\sigma_c =$

Continuum with sharp crack
$$2a$$

 $\sigma_f = \sqrt{\frac{E\gamma_s}{4a}} \Rightarrow$

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Griffith's verification experiment

 Glass fibers with artificial cracks (much larger than natural crack-like flaws), tension tests

| | Crack Length, 2 <i>a</i> | Measured Strength, σ | $\sigma_f \sqrt{a}$ | |
|-------------------------------------|--------------------------|--|--|--|
| | mm | MPa | MPa \sqrt{m} | |
| sample 1 | 3.8 | 6.0 | 0.26 | |
| sample 2 | 6.9 | 4.3 | 0.25 | |
| sample 3 | 13.7 | 3.3 | 0.27 | |
| sample 4 | 22.6 | 2.5 | 0.27 | |
| (Data from the Griffith experiment) | | | | |
| | | $\sigma_f = \sqrt{\frac{E\gamma_s}{4a}}$ | $\sigma_f \sqrt{a} = \sqrt{\frac{E\gamma_s}{4}} = \text{const.}$ | |

4.1.3. Cause of discrepancy:

2. Energy approach



It indicates that the work rate supplied to the continuum by the applied loads is equal to the rate of the elastic strain energy and plastic strain work plus the energy dissipated in crack propagation



Comparison of stress & energy approaches

Stress approach: Stress Concentration

$$\sigma_f = 0.5 \sqrt{\frac{E\gamma_s}{a}}$$

Energy approach: Griffith $\sigma_f = \sqrt{\frac{2}{\pi}} \sqrt{\frac{E\gamma_s}{a}} \approx 0.8 \sqrt{\frac{E\gamma_s}{a}}$

Energy equation for ductile materials

$$\sigma_c = \sqrt{\frac{2E\gamma_s}{\pi a}}$$

Griffith (1921), ideally brittle solids

$$\sigma_c = \sqrt{\frac{2E(\gamma_s + \gamma_p)}{\pi a}}$$

Irwin, Orowan (1948), metals

 γ_p plastic work per unit area of surface created

 $\gamma_p \gg \gamma_s$

 $\gamma_p \approx 10^3 \gamma_s$ (metals)
Generalization of Energy equation $\sigma_f = \sqrt{\frac{2Ew_f}{\pi a}}$

- w_f : Fracture energy from plastic, viscoelastic, or viscoplastic effects
- w_f can also be influenced by crack meandering and branching
- Caution: If nonlinear displacement regions are large enough this equation is not accurate as it is based on linear elastic solution $(\Pi = \Pi_0 \frac{\pi \sigma^2 a^2 B}{E})$



Energy release rate

Irwin 1956



a.k.a Crack extension force Crack driving force

G: Energy released during fracture per unit of newly created fracture surface area



energy available for crack growth (crack driving force)

Energy release rate *failure criterion*

 $\fbox{G \ge G_c}$

fracture energy, considered to be a material property (independent of the

applied loads and the geometry of the body).

Strain energy density

Consider a linear elastic bar of stiffness k, length L, area A, subjected to a force F, the work is

This work will be completely stored in the structure in the form of <u>strain energy</u>. Therefore, the external work and strain energy are equal to one another

In terms of stress/strain

$$U = W = \frac{1}{2}Fu$$

$$U = \frac{1}{2}Fu = \frac{1}{2}\frac{F}{A}\frac{u}{L}AL$$

$$U = \frac{1}{2}Fu = \frac{1}{2}\frac{F}{A}\frac{u}{L}AL$$

$$\varepsilon_x$$
Strain energy density
$$u = \frac{1}{2}\sigma_x \epsilon_x \longrightarrow u = \int \sigma_x d\epsilon_x$$

Strain energy density $u = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \underbrace{\nu}_E \sigma_x \sigma_y + \sigma_y \sigma_z$ $+ \sigma_z \sigma_x) + \frac{1}{2\mu}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)$

Plane problems $\mu = rac{E}{2(1+
u)}$ shear modulus

$$u = \frac{1}{4\mu} \left[\frac{\kappa + 1}{4} (\sigma_x^2 + \sigma_y^2) - 2(\sigma_x \sigma_y - \tau_{xy}^2) \right]$$

Kolosov coefficient

 $\kappa = \begin{cases} 3 - 4\nu & \text{plane strain} \\ \frac{3 - \nu}{1 + \nu} & \text{plane stress} \end{cases}$

Given: A point load - displacement system $P
ightarrow n_1 : a$ with a crack and two data points: $P
ightarrow n_2 : a$

- Load P_1 , displacement u_1 , & crack length a_1
- Load P_2 , displacement u_2 , & crack length $a_2 = a_1 + \Delta a \pmod{\Delta a}$

Goal: Compute GNotation:

- W_{12} : External work from n_1 to n_2
- W_{o1} External work that would have happened through elastic (or almost elastic) deformation with fixed crack length from 0 to n_1 .
- W_{o2} Similar to W_{o1}



Questions:

- What is the value of $W_1 = W_{o1}$?
- For two paths from *o* to 2 (02 and 012) can we write:

 $W_{o2} = W_{o1} + W_{12} \implies W_{12} = W_{o2} - W_{o1}?$

Explain.

• How G is calculated from the figure?



• What is the value of $W_1 = W_{o1}$?

$$W_1 = W_{o1} = \frac{1}{2}P_1u_1 = U_{e1}$$

Linear system for fixed crack length a from o to n_1 .

• For two paths from *o* to 2 (02 and 012) can we write:

$$W_{o2} = W_{o1} + W_{12} \implies W_{12} = W_{o2} - W_{o1}?$$

No. Because this is a dissipative mechanism between n_1 and n_2 due to crack growth and work is path dependent.



$$G = -\frac{d\Pi}{dA} = -\frac{d(U_e - W)}{dBa} P \bigwedge_{n_2 : a + \Delta a}^{n_1 : a} \frac{n_2 : a + \Delta a}{n_2 : a + \Delta a}$$

$$\approx -\frac{1}{B\Delta a} (\Delta U_e - \Delta W) P_1 \bigwedge_{n_2 : a + \Delta a}^{n_1 : a} \frac{n_1}{n_2 : a + \Delta a}$$

$$P_1 \bigwedge_{n_2 : a + \Delta a}^{n_2 : a + \Delta a} P_1 \bigwedge_{n_2 : a + \Delta a}^{n_2 : a + \Delta a} \frac{n_1}{n_2} \sum_{n_2 : a + \Delta a}^{n_2 : a + \Delta a} \frac{n_1}{n_2} \sum_{n_2 : a + \Delta a}^{n_2 : a + \Delta a} \frac{n_1}{n_2} \sum_{n_2 : a + \Delta a}^{n_2 : a + \Delta a} \sum_{n_2 : a +$$

G from experiments



u

$$G(a_3) = \frac{1}{B} \frac{\text{shaded area}}{a_4 - a_3}$$

G from experiment

B: thickness

 $\Pi = U_e - W$

a1: OA, triangle OAC=U

a₂: OB, triangle OBC=U

Fixed grips

$$\Delta U_e = -\frac{1}{2}(P_1 - P_2)u$$
$$\Delta W = 0 \quad \Rightarrow$$
$$G = \frac{1}{B\Delta a} \frac{1}{2}(P_1 - P_2)u$$

$$G = \frac{1}{B} \frac{(OAB)}{\Delta a}$$

Crack can grow from A to B or we obtain OA, OB with two different tests with different a values.



G from experiment

a1: OA, triangle OAC=U

a₂: OB, triangle OBC=U

Dead loads



B: thickness $\Pi = U_e - W$

 $\Delta U_e = \frac{1}{2} P(u_2 - u_1),$ $\Delta W = P(u_2 - u_1) \Rightarrow$ $G = \frac{1}{B\Delta a} \frac{1}{2} P(u_2 - u_1)$ OAB=ABCD-(OBD-OAC)

 $G = \frac{1}{B} \frac{(OAB)}{\Delta a}$

Crack can grow from A to B or we obtain OA, OB with two different tests with different a values.

W = 0

0





G in terms of compliance Fixed grips Fixed loads

$$G = \frac{1}{2B} \frac{u^2}{C^2} \frac{dC}{da} = \frac{1}{2B} P^2 \frac{dC}{da} \qquad \qquad G = \frac{1}{2B} P^2 \frac{dC}{da}$$

Strain energy release rate is identical for fixed grips and fixed loads.

Strain energy release rate is proportional to the differentiation of the compliance with respect to the crack length.

Crack extension resistance curve (R-curve)



α

R-curve shapes





Double cantilever beam (DCB) example: load control



Double cantilever beam (DCB) example: displacement control



Example

The following data were obtained from a series of tests conducted on precracked specimens of thickness 1 mm.

| Crack length a(mm) | Critical load P(kN) | Critical displacement u(mm) | | |
|-----------------------|------------------------|--------------------------------|--|--|
| 30.0 | 4.00 | 0.40 | | |
| 40.0 | 3.50 | 0.50 | | |
| 50.5 | 3.12 | 0.63 | | |
| 61.6 | 2.80 | 0.78 | | |
| 71.7 | 2.62 | 0.94 | | |
| 79.0 | 2.56 | 1.09 | | |

where P and u are the critical load and displacement at crack growth. The loaddisplacement record for all crack lengths is linearly elastic up to the critical point. Determine the critical value of the strain energy release rate $G_c = R$ from: (a) the load-displacement records, and (b) the compliance-crack length curve.



| Area | OA_1A_2 | OA_2A_3 | OA_3A_4 | OA_4A_5 | OA_5A_6 |
|-----------------------------|-----------|-----------|-----------|-----------|-----------|
| $G_{\rm c} = R(\rm kJ/m^2)$ | 30.0 | 30.7 | 30.2 | 29.1 | 30.8 |

Gc for different crack lengths are almost the same: flat Rcurve. 102

For the determination of $G_c = R$ from the compliance-crack length curve we first determine the following values of compliance C = u/P for the various crack lengths

| a(mm) | 30.0 | 40.0 | 50.5 | 61.6 | 71.7 | 79.0 |
|---------------------------------|------|------|------|------|------|------|
| $C(\times 10^{-7} \text{ m/N})$ | 1.00 | 1.43 | 2.02 | 2.79 | 3.59 | 4.26 |

P = 4 kN

$$\frac{\mathrm{d}C}{\mathrm{d}a} = \frac{(1.43 - 1.00) \times 10^{-7} \,\mathrm{m/N}}{10 \times 10^{-3} \,\mathrm{m}} = 4.3 \times 10^{-6} \,\mathrm{N^{-1}}$$

and

$$G_c = R = \frac{(4 \times 10^3)^2 \text{ N}^2 \times (4.3 \times 10^{-6} \text{ N}^{-1})}{2 \times 10^{-3} \text{ m}} = 34.4 \text{ kJ/m}^2$$



For the crack lengths a_2 , a_3 , a_4 and a_5 , dC/da can be determined as the mean value of the left and right derivatives of C. For example, for the crack length a = 50.5 mm, we have

 $P = 3.12 \, \text{kN}$

$$\left(\frac{\mathrm{d}C}{\mathrm{d}a}\right)_{l} = \frac{(2.02 - 1.43) \times 10^{-7} \mathrm{m/N}}{10.5 \times 10^{-3} \mathrm{m}} = 5.6 \times 10^{-6} \mathrm{N}^{-1}$$
$$\left(\frac{\mathrm{d}C}{\mathrm{d}a}\right)_{r} = \frac{(2.77 - 2.02) \times 10^{-7} \mathrm{m/N}}{11.1 \times 10^{-3} \mathrm{m}} = 6.8 \times 10^{-6} \mathrm{N}^{-1}$$

4.2. Stress solutions, Stress Intensity Factor K (SIF)

Elastodynamics Boundary value problem Kinematics



Elastostatics Boundary value problem **Constitutive equation** Hook's law $\sigma_{ij} = D_{ijkl}\varepsilon_{kl} \qquad i, j, k, l = 1, 2, 3$ Isotropic 3D $T_{ij} = \lambda \delta_{ij} E_{kk} + 2\mu E_{ij}$ or $\mathbf{T} = \lambda \mathbf{I}_E + 2\mu \mathbf{E}$ 2D (plane strain) $\begin{cases} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau \end{cases} = \frac{E}{(1+\nu)(1-2\nu)} \begin{vmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ \nu & \nu & 0 \\ 0 & 0 & 1-2\nu \end{vmatrix} \begin{cases} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{cases}$ **2D** (plane stress) $\begin{cases} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{cases} = \frac{1}{1-\nu^2} \begin{vmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{\nu} \end{vmatrix} \begin{cases} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma \end{cases}$ $p = \rho \mathbf{v}$ **Balance of linear momentum** $\nabla .\sigma + \rho \mathbf{b} = \dot{p}$ $\nabla \sigma + \rho \mathbf{b} = \dot{p}$ or $\sigma_{ij,j} + \rho b_j = \rho \ddot{u}_i$ $\geq \partial \mathcal{D}_f$ $\bar{\mathbf{t}}$ specified Static: p
 = 0 **ū** specified No body force b = 0 ∂D :

 x_1

Displacement approach



PDE + BC for $\mathbf{u} \Rightarrow \epsilon \Rightarrow \sigma$

Stress function approach

What are Airy stress function approach?

Use of stress function \Rightarrow

Balance of linear momentum is automatically satisfied (no body force, static)

$$\psi(x_1, x_2) \quad \rightarrow \quad \sigma_{ij} = -\psi_{,ij} + \delta_{ij}\psi_{,kk} \quad \rightarrow \quad \sigma_{ij,j} = 0$$

How do we obtain strains and displacements?

Strain: compliance *e.g.*

$$\varepsilon_{ij} = \frac{1+\nu}{E} \left(\sigma_{ij} - \nu \delta_{ij} \sigma_{kk} \right)$$
$$\epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$$

Can we always obtain u by integration? No

3 displacements (unknowns) 6 strains (equations)

Displacements: Integration of

Need to satisfy strain compatibility condition(s)

$$\frac{\partial^2 \varepsilon_{ik}}{\partial x_j \partial x_j} + \frac{\partial^2 \varepsilon_{jj}}{\partial x_i \partial x_k} - \frac{\partial^2 \varepsilon_{jk}}{\partial x_i \partial x_j} - \frac{\partial^2 \varepsilon_{ij}}{\partial x_j \partial x_k} = 0.$$

Stress function approach $\varepsilon_{ij} = \frac{1+\nu}{E} \left\{ -\psi_{,ij} + (1-\nu)\delta_{ij}\psi_{,kk} \right\}$ $2\varepsilon_{12,12} - \varepsilon_{11,22} - \varepsilon_{22,11} = 0$ $2\psi_{,1122} + \psi_{,2222} + \psi_{,1111} = 0$ $(\psi_{,11} + \psi_{,22})_{,11} + (\psi_{,11} + \psi_{,22})_{,22} = 0$ Laplace operator : $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = ()_{11} + ()_{22}$ \rightarrow bi-harmonic equation bi-harmonic equation $\nabla^2(\nabla^2\psi) = \nabla^4\psi = 0$ BC's BC's Stress function approach: Displacement approach: $PDE + BC \text{ for } \mathbf{u} \Rightarrow \epsilon \Rightarrow \sigma$ $N \Rightarrow \sigma \Rightarrow \epsilon \Rightarrow \mathbf{u}$

- Generally no need to solve biharmonic function: Extensive set of functions from complex analysis
- No need to solve any PDE
- Only working with the scalar stress function

Complex numbers

- Complex numbers
 - $z = x_1 + ix_2 = \operatorname{Re}(z) + i\operatorname{Im}(z)$ $\operatorname{Re}(z) = x_1$ $\operatorname{Im}(z) = x_2$ $z = re^{i\theta}$ $\bar{z} = x_1 ix_2 = re^{-i\theta} \text{ conjugate}$
- Complex functions

$$\begin{split} f(z) &= \operatorname{Re} f(z) + i \operatorname{Im} f(z) = U(x_1, x_2) + i V(x_1, x_2) \\ & U \& V : \text{conjugate harmonic functions: } \nabla^2 U = \nabla^2 V = 0 \end{split}$$



• Derivative relations

$$\frac{\partial f(z)}{\partial x_1} = \frac{\partial f(z)}{\partial z} \frac{\partial z}{\partial x_1} = f'(z)$$
$$\frac{\partial f(z)}{\partial x_2} = \frac{\partial f(z)}{\partial z} \frac{\partial z}{\partial x_2} = if'(z)$$
$$\frac{\partial \operatorname{Re} f(z)}{\partial x_1} = \operatorname{Re} f'(z)$$
$$\frac{\partial \operatorname{Im} f(z)}{\partial x_1} = \operatorname{Im} f'(z)$$
$$\frac{\partial \operatorname{Re} f(z)}{\partial x_2} = -\operatorname{Im} f'(z)$$
$$\frac{\partial \operatorname{Im} f(z)}{\partial x_2} = \operatorname{Re} f'(z)$$



Stress function approach

• Any biharmonic solution can be expressed by Kolonov-Muskhelishvili complex potentials, ϕ, χ :

$$\Psi(x_1, x_2) = \operatorname{Re}\left[\bar{z}\phi + \chi\right]$$

• Stresses are obtained differentiation,

$$\sigma_{11} = \Psi_{,22} = \operatorname{Re} \left[\phi' - \frac{1}{2} \bar{z} \phi'' - \frac{1}{2} \chi'' \right]$$

$$\sigma_{22} = \Psi_{,11} = \operatorname{Re} \left[\phi' + \frac{1}{2} \bar{z} \phi'' + \frac{1}{2} \chi'' \right]$$

$$\sigma_{12} = -\Psi_{,12} = \frac{1}{2} \operatorname{Re} \left[\bar{z} \phi'' + \chi'' \right]$$

• Displacements are obtained by integration of strains:

$$u_{1} = \operatorname{Re} \left[\kappa \phi - \bar{z} \phi' - \chi' \right]$$
$$u_{2} = \operatorname{Im} \left[\kappa \phi + \bar{z} \phi' + \chi' \right]$$
$$\kappa = \begin{cases} 3 - 4\nu & \text{plane strain} \\ \frac{3 - \nu}{1 + \nu} & \text{plane stress} \end{cases}$$

Crack modes



Mode I: Opening Mode II: In-plane shear Mode III: Out-of-plane she**ar**







Westergaard's complex stress function for mode I

• Constructing appropriate stress function (ignoring const. parts):

$$\begin{array}{l} 0 = \sigma_{12}(x_1, x_2 = 0) \quad \Rightarrow \\ \chi'' = -\bar{z}\phi'' = -z\phi'' \; (x_2 = 0) \quad \Rightarrow \\ \chi' = -z\phi' + \phi \quad \Rightarrow \\ \chi = -z\phi + 2\tilde{\phi} \; (\{\tilde{\phi}\}' = \phi) \quad \Rightarrow \\ \Psi = \operatorname{Re}\left[\bar{z}\phi + \chi\right] = \operatorname{Re}\left[\bar{z}\phi - z\phi + 2\tilde{\phi}\right] \\ = 2\operatorname{Re}\left[-ix_2\phi + \tilde{\phi}\right] = 2\left[x_2\operatorname{Im}\phi + \operatorname{Re}\tilde{\phi}\right] \\ \tilde{Z} = \int f(z)\mathrm{d}z, \; \widetilde{\widetilde{Z}} = \int \tilde{Z}(z)\mathrm{d}z \\ \tilde{Z} \text{ and } \; \widetilde{\widetilde{Z}} \text{ are 1st, 2nd antiderivatives} \\ \operatorname{Using} \; \phi = \frac{1}{2}\widetilde{Z} \end{array}$$





Westergaard's complex stress function for mode I

y ImZ'

y ImZ'

$$Z(z), z = x + iy, i^{2} = -1$$

$$\Psi = \operatorname{Re} \tilde{Z}^{\kappa} + y \operatorname{Im} \tilde{Z}$$

$$\tilde{Z} = \int f(z) dz, \tilde{Z}^{\kappa} = \int \tilde{Z}(z) dz$$

$$\kappa$$

$$Kolosov \operatorname{coef.} \kappa$$

$$\kappa = \begin{cases} 3 - 4\nu & \text{plane strain} \\ \frac{3 - \nu}{1 + \nu} & \text{plane stress} \end{cases}$$

$$\mu = \frac{E}{2(1 + \nu)} \text{ shear modulus}$$

$$u = \frac{\kappa + 1}{2} \operatorname{Im} \tilde{Z} - y \operatorname{Re} Z$$

Griffith's crack (mode I)

$$(x,y) \rightarrow \infty : \sigma_{xx} = \sigma_{yy} = \sigma, \tau_{xy} = 0$$

$$|x| < a, y = 0 : \sigma_{yy} = \tau_{xy} = 0$$
boundary conditions
$$\int_{a}^{a} \int_{a}^{b} \int_{a}^{b} \int_{a}^{b} \int_{a}^{c} \int_{a}^{c}$$

 $(x,y) \rightarrow \infty : z \rightarrow \infty \quad Z \rightarrow \sigma$ 118
Griffith's crack (mode I)

$$(x,y) \to \infty : \sigma_{xx} = \sigma_{yy} = \sigma, \tau_{xy} = 0$$

$$|x| < a, y = 0 : \sigma_{yy} = \tau_{xy} = 0$$

$$(Z(z) = \frac{\sigma z}{\sqrt{z^2 - a^2}})$$





$$\begin{split} \xi &= z-a, \xi = r e^{i\theta} \\ Z(z) &= \frac{\sigma(\xi+a)}{\sqrt{\xi(\xi+2a)}} \end{split}$$

Griffith's crack (mode I)

$$Z(z) = \frac{\sigma(\xi + a)}{\sqrt{\xi(\xi + 2a)}} = \frac{\sigma(\xi + a)}{\sqrt{2a\xi(1 + \xi/(2a)))}}$$

$$\sqrt{1 + \xi/(2a)} = (1 + \xi/(2a))^{-1/2}$$

$$= 1 - \frac{1}{2}\frac{\xi}{2a} + \text{H.O.T}$$

$$= 1 - \frac{1}{2}\frac{\xi}{2a} + \text{H.O.T}$$

 ξ small $\xi + a = a$

$$\longrightarrow \quad Z = \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi\xi}}$$

$$Z(z) = \frac{K_{I}}{\sqrt{2\pi\xi}}, K_{I} = \sigma\sqrt{\pi a}$$

$$Z(z) = \frac{K_{I}}{\sqrt{2\pi\tau}} e^{-i\theta/2} \qquad \xi = re^{i\theta}$$

$$Z'(z) = -\frac{1}{2} \frac{K_{I}}{\sqrt{2\pi}} e^{-3/2} = -\frac{K_{I}}{2r\sqrt{2\pi\tau}} e^{-i3\theta/2}$$

$$Z'(z) = -\frac{1}{2} \frac{K_{I}}{\sqrt{2\pi}} \xi^{-3/2} = -\frac{K_{I}}{2r\sqrt{2\pi\tau}} e^{-i3\theta/2}$$

$$e^{-ix} = \cos x - i \sin x$$

$$y = r \sin \theta$$

$$\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$
inverse square root
$$\sigma_{yy} = \frac{K_{I}}{\sqrt{2\pi\tau}} \cos\left(\frac{\theta}{2}\right) \left[1 + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\tau_{yy} = \frac{K_{I}}{\sqrt{2\pi\tau}} \cos\left(\frac{\theta}{2}\right) \left[1 + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\tau_{yy} = \frac{K_{I}}{\sqrt{2\pi\tau}} \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right)$$

$$r \to 0: \sigma_{ij} \to \infty$$

Plane strain problems

Hooke's law

$$\epsilon_{zz} = \frac{1}{E} \left(-\nu \sigma_{xx} - \nu \sigma_{yy} + \sigma_{zz} \right)$$

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{\theta}{2} \right)$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \sin \frac{\theta}{2} \sin \frac{\theta}{2} \sin \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \sin \frac{\theta}{$$

$$\longrightarrow \sigma_z = \nu(\sigma_x + \sigma_y)$$

$$F_1$$
 F_2 F_2 F_2

$$\sigma_z = 2\nu \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2}$$

 3θ

 $\mathbf{2}$

 3θ

Stresses on the crack plane

$$\begin{aligned} \sigma_{xx} &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \cos \frac{3\theta}{2} \right) \\ \sigma_{yy} &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \cos \frac{3\theta}{2} \right) \\ \tau_{xy} &= \frac{K_I}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \sin \frac{3\theta}{2} \end{aligned}$$

On the crack plane

 $\theta = 0, r = x$ $\sigma_{xx} = \sigma_{yy} = \frac{K_I}{\sqrt{2\pi x}}$ $\sigma_{xx} = \sigma_{yy} = \frac{K_I}{\sqrt{2\pi x}}$ $\sigma_{xy} = 0$ $\sigma_{xy} = \sigma_{yy}$ $\sigma_{1} = \sigma_{2} = \sigma_{xx} = \sigma_{yy}$

Stress Intensity Factor (SIF)

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right)\right]$$
$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 + \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right)\right]$$
$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right)$$

$$K_I = \sigma \sqrt{\pi a}$$

 $[MPa\sqrt{m}]$

- Stresses-K: linearly proportional
- K uniquely defines the crack tip stress field
- modes I, II and III:
- LEFM: single-parameter

 K_I, K_{II}, K_{III}





Mode II problem

Check BCs

τ

Boundary conditions

$$(x,y) \to \infty : \sigma_{xx} = \sigma_{yy} = 0, \tau_{xy} = \tau$$
$$|x| < a, y = 0 : \sigma_{yy} = \tau_{xy} = 0$$
$$\left(Z = -\frac{i\tau z}{\sqrt{z^2 - a^2}}\right)$$



Stress function

$$\sigma_{yy} = \operatorname{Re}Z + y \operatorname{Im}Z'$$

$$\tau_{xy} = -y \operatorname{Re}Z'$$

Mode II problem



Stress function

$$\begin{aligned}
(x,y) \to \infty : \sigma_{xx} = \sigma_{yy} = 0, \tau_{xy} = \tau \\
|x| < a, y = 0 : \sigma_{yy} = \tau_{xy} = 0
\end{aligned}$$

$$\begin{aligned}
(x,y) \to \infty : \sigma_{xx} = \sigma_{yy} = 0, \tau_{xy} = \tau \\
|x| < a, y = 0 : \sigma_{yy} = \tau_{xy} = 0
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(x|< a, y = 0 : \sigma_{yy} = \tau_{xy} = 0
\end{aligned}$$

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(x|< a, y = 0 : \sigma_{yy} = \tau_{xy} = 0
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(x,y) \to \infty : \sigma_{xx} = \sigma_{yy} = 0, \tau_{xy} = \tau \\
(x|< a, y = 0 : \sigma_{yy} = \tau_{xy} = 0
\end{aligned}$$

$$\begin{aligned}
(x,y) \to \infty : \sigma_{xx} = \sigma_{yy} = 0, \tau_{xy} = \tau \\
(x|= -\frac{i\tau z}{\sqrt{z^2 - a^2}}, \tau_{xy} = 0
\end{aligned}$$

$$\begin{aligned}
(x,y) \to \infty : \sigma_{xx} = \sigma_{yy} = 0, \tau_{xy} = \tau \\
(x|= -\frac{i\tau z}{\sqrt{z^2 - a^2}}, \tau_{xy} = 0
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(x,y) \to \infty : \sigma_{xy} = \sigma_{xy} = 0, \tau_{xy} = \tau \\
(x|= -\frac{i\tau z}{\sqrt{z^2 - a^2}}, \tau_{xy} = 0, \tau_{xy} = 0
\end{aligned}$$

$$\begin{aligned}
(x,y) \to \infty : \sigma_{xy} = \sigma_{xy} = 0, \tau_{xy} = \tau \\
(x|= -\frac{i\tau z}{\sqrt{z^2 - a^2}}, \tau_{xy} = 0, \tau_{xy$$

$$K_{II} = \tau \sqrt{\pi a}$$

mode II SIF

Universal nature of the asymptotic stress field Westergaards, Sneddon etc.

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right)\right]$$
$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 + \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right)\right]$$
$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right)$$

(mode I)

$$\sigma_{xx} = -\frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \left(2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right)$$
$$\sigma_{yy} = \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2}$$
$$\tau_{xy} = \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)$$

(mode II)

Irwin
$$\sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta) + \text{H.O.T}$$

$$\begin{aligned} & \text{Mode I: displacement field} \\ & Z(z) = \frac{K_I}{\sqrt{2\pi r}} \left(\cos \frac{\theta}{2} - i \sin \frac{\theta}{2} \right) \\ & Z(z) = \frac{K_I}{\sqrt{2\pi \xi}} \quad \bar{Z} = \int Z(z) dz \end{aligned} \begin{aligned} & 2\mu u = \frac{\kappa - 1}{2} \operatorname{Re} \tilde{Z} - y \operatorname{Im} Z \\ & 2\mu v = \frac{\kappa + 1}{2} \operatorname{Im} \tilde{Z} - y \operatorname{Re} Z \end{aligned} \\ & \tilde{Z}(z) = 2 \frac{K_I}{\sqrt{2\pi}} \xi^{1/2} = 2K_I \sqrt{\frac{r}{2\pi}} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right) \qquad z = \xi + a \\ & \xi = r e^{i\theta} \end{aligned} \\ & e^{-ix} = \cos x - i \sin x \end{aligned} \\ & u = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left(\kappa - 1 + 2 \sin^2 \frac{\theta}{2} \right) \end{aligned} \\ & \kappa = \begin{cases} 3 - 4\nu & \text{plane strain} \\ \frac{3 - \nu}{1 + \nu} & \text{plane strain} \\ \end{array} \end{aligned}$$

Crack face displacement

 $y=0, -a \leq x \leq a$





$$-a \le x \le a \quad i = \sqrt{-1} \qquad \longrightarrow \tilde{Z}(z) = i(\sigma\sqrt{a^2 - x^2})$$

ellipse

v

COD

$$v = \frac{\kappa + 1}{4\mu} \sigma \sqrt{a^2 - x^2}$$

Crack Opening Displacement

$$\text{COD} = 2v = \frac{\kappa + 1}{2\mu} \sigma \sqrt{a^2 - x_{_{130}}^2}$$

Crack tip stress field in polar coordinates-mode I



Principal crack tip stresses

$$\sigma_{1}, \sigma_{2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^{2} + \tau_{xy}^{2}}$$

$$\sigma_{1}, \sigma_{2} = \frac{K_{I}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2}\right)$$

$$\sigma_{2} = \frac{K_{I}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2}\right)$$

$$\sigma_{3} = \begin{cases} 0 & \text{plane stress} \\ \frac{2\nu K_{I}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} & \text{plane strain} \\ \frac{2\nu K_{I}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} & \text{plane strain} \end{cases}$$

$$\leftarrow \sigma_{3} = \nu(\sigma_{1} + \sigma_{2})$$

Crack solution using V-Notch

• Airy stress function assumed to be

$$\Phi(r,\theta) \equiv r^{\lambda+1}F(\theta,\lambda)$$

• Compatibility condition

$$\nabla^2 \left(\nabla^2 \Phi \right) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} \right) = 0 \implies$$

$$\frac{\partial^4 F(\theta,\lambda)}{\partial \theta^4} + 2(\lambda^2 + 1)\frac{\partial^2 F(\theta,\lambda)}{\partial \theta^2} + (\lambda^2 - 1)^2 F(\theta,\lambda) = 0 \implies$$

$$F(\theta) = e^{m\theta} \Rightarrow$$

$$\left[(1-\lambda)^2 + m^2 \right] \left[(1+\lambda)^2 + m^2 \right] = 0 \Rightarrow$$

$$m = \pm i(1\pm\lambda) \Rightarrow$$



Source: Saouma:2010 Boulder Fracture Mechanics Williams, 1952

• Final form of stress function

$$\Phi(r,\theta) = r^{\lambda+1} \underbrace{\left[A\cos(\lambda-1)\theta + B\cos(\lambda+1)\theta + C\sin(\lambda-1)\theta + D\sin(\lambda+1)\theta\right]}_{F(\theta,\lambda)}$$

Crack solution using V-Notch

• Stress values

$$\sigma_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2} = r^{\lambda-1}\lambda(\lambda+1)F(\theta)$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right) = r^{\lambda-1}[-\lambda F'(\theta)$$

• Boundary conditions

$$\begin{array}{rcl} \sigma_{\theta\theta} \mid_{\theta=\pm\alpha} &=& 0\\ \sigma_{r\theta} \mid_{\theta=\pm\alpha} &=& 0 \end{array} \Longrightarrow$$



$$F(\alpha) = F(-\alpha) = F'(\alpha) = F'(-\alpha) = 0 \implies$$

$$\begin{bmatrix} \cos(\lambda - 1)\alpha & \cos(\lambda + 1)\alpha & 0 & 0\\ \omega \sin(\lambda - 1)\alpha & \sin(\lambda + 1)\alpha & 0 & 0\\ 0 & 0 & \sin(\lambda - 1)\alpha & \sin(\lambda + 1)\alpha\\ 0 & 0 & \omega \cos(\lambda - 1)\alpha & \cos(\lambda + 1)\alpha \end{bmatrix} \begin{cases} A\\ B\\ C\\ D \end{cases} = 0$$

$$\omega = \frac{\lambda - 1}{\lambda + 1}.$$

• Eigenvalues and eigenvectors for nontrivial solutions

$$\sin 2\lambda_n \alpha + \lambda_n \sin 2\alpha = 0 \quad \longleftarrow \quad \text{Mode I}$$
$$\sin 2\xi_n \alpha - \xi_n \sin 2\alpha = 0 \quad \longleftarrow \quad \text{Mode II}$$

Crack solution using V-Notch

• Sharp crack

$$\alpha = \pi \implies \frac{\sin(2\pi\lambda_n) = 0}{\sin(2\pi\xi_n) = 0} \implies \lambda_n = \frac{n}{2} \text{ with } n = 1, 3, 4, \dots \text{ (n = 2 constant stress)}$$

• After having eigenvalues, eigenvectors are obtained from

Mode I $\begin{array}{rcl} A_n \cos(\lambda_n - 1)\alpha + B_n \cos(\lambda_n + 1)\alpha &=& 0\\ A_n \omega \sin(\lambda_n - 1)\alpha + B_n \sin(\lambda_n + 1)\alpha &=& 0 \end{array}$

Mode II $C_n \sin(\xi_n - 1)\alpha + D_n \sin(\xi_n + 1)\alpha = 0$ $C_n \omega \cos(\xi_n - 1)\alpha + D_n \cos(\xi_n + 1)\alpha = 0$

• First term of stress expansion

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{5}{4} \cos \frac{\theta}{2} - \frac{1}{4} \cos \frac{3\theta}{2} \right) + \frac{K_{II}}{\sqrt{2\pi r}} \left(-\frac{5}{4} \sin \frac{\theta}{2} + \frac{3}{4} \sin \frac{3\theta}{2} \right) \right]$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{3}{4} \cos \frac{\theta}{2} + \frac{1}{4} \cos \frac{3\theta}{2} \right) + \frac{K_{II}}{\sqrt{2\pi r}} \left(-\frac{3}{4} \sin \frac{\theta}{2} - \frac{3}{4} \sin \frac{3\theta}{2} \right) \right]$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{1}{4} \sin \frac{\theta}{2} + \frac{1}{4} \sin \frac{3\theta}{2} \right) + \frac{K_{II}}{\sqrt{2\pi r}} \left(\frac{1}{4} \cos \frac{\theta}{2} + \frac{3}{4} \cos \frac{3\theta}{2} \right) \right]$$



Inclined crack in tension



Cylindrical pressure vessel with an inclined through-thickness crack



 $rac{R}{t} \geq 10$ thin-walled pressure π $(\pi R^2)p = (2\pi Rt)\sigma_z$



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 $(\Delta l 2R)p = (2\Delta l t)\sigma_{\theta}$



Cylindrical pressure vessel with an inclined through-thickness crack



Computation of SIFs

- Analytical methods (limitation: simple geometry)
 - superposition methods
 - weight/Green functions
- Numerical methods (FEM, BEM, XFEM)

numerical solutions -> data fit -> SIF handbooks

- Experimental methods
 - photoelasticity

SIF for finite size samples

Exact (closed-form) solution for SIFs: simple crack geometries in an **infinite** plate.

Cracks in finite plate: influence of external boundaries cannot be neglected -> generally, no exact solution



SIFs handbook





$$K_{I} = \sqrt{\sec \frac{\pi a}{W}} \sigma \sqrt{\pi a}$$

secant function
$$\sec \theta = \frac{1}{\cos \theta}$$

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SIFs handbook



SIFs handbook



Superposition method

A sample in mode I subjected to tension and bending:

$$\sigma_{ij} = \frac{K_I^{\text{tension}}}{\sqrt{2\pi r}} f_{ij}(\theta) + \frac{K_I^{\text{bending}}}{\sqrt{2\pi r}} f_{ij}(\theta)$$

$$\sigma_{ij} = \frac{K_I^{\text{tension}} + K_I^{\text{bending}}}{\sqrt{2\pi r}} f_{ij}(\theta) \xrightarrow{P} \underbrace{\left(\bigcup_{M} \sqrt{\frac{1}{2}} \prod_{M} \frac{1}{\sqrt{2\pi r}} \right)}_{M} \xrightarrow{P}} K_I = K_I^{\text{tension}} + K_I^{\text{bending}}$$

Is superposition of SIFs of different crack modes possible?

Determine the stress intensity factor for an edge cracked plate subjected to a combined tension and bending.



Solution

$$\begin{split} K_I^{\text{bend}} &= f_M(a/W) \frac{6M}{BW^2} \sqrt{\pi a} \qquad K_I^{\text{ten}} = f_P(a/W) \frac{P}{BW} \sqrt{\pi a} \\ \uparrow \\ 1.055 \qquad 1.12 \end{split}$$

$$K_I = \left(1.055 \frac{6M}{BW^2} + 1.12 \frac{P}{BW}\right) \sqrt{\pi a}$$

Superposition method (cont.)

Centered crack under internal pressure





 $K_I = \sigma \sqrt{\pi a}$

SIFs: asymmetric loadings

Procedure: build up the case from symmetric cases and then to subtract the superfluous loadings.



$$K_A = K_B + K_C - K_D$$

$$K_A = (K_B + K_C)/2$$

Note: The circle should be in the middle of the plate

Circle must be in the middle of domain (both vertically and horizontally)

Two small cracks at a hole



$$K_{I} = 1.12(3\sigma)\sqrt{\pi(a-R)} = 3.36\sigma\sqrt{\pi a}\sqrt{1 - \frac{R}{a}} = 3.36\sqrt{1 - \frac{1}{a/R}}\sigma\sqrt{\pi a}$$

Photoelasticity

Wikipedia

Photoelasticity is an experimental method to <u>determine the stress distribution</u> in a material. The method is mostly used in cases where mathematical methods become quite cumbersome. Unlike the analytical methods of stress determination, photoelasticity gives a fairly accurate picture of stress distribution, even around abrupt discontinuities in a material. The method is an important tool for determining critical stress points in a material, and is used for determining stress concentration in irregular geometries.



K-G relationship

So far, two parameters that describe the behavior of cracks: K and G.

K: local behavior (tip stresses) G: global behavior (energy)



Crack closure analysis: work to open the crack = work to close the crack



u_v



K-G relationship (cont.)

Mode I

$$G_{I} = \begin{cases} \frac{K_{I}^{2}}{E} & \text{plane stress} \\ (1-v^{2}) \frac{K_{I}^{2}}{E} & \text{plane strain} \end{cases}$$

Mixed mode

$$G = rac{K_I^2}{E'} + rac{K_{II}^2}{E'} + rac{K_{III}^2}{2\mu}$$
 $E' = \left\{ egin{array}{c} rac{E}{1-
u^2} & ext{for plane strain} \ E & ext{for plane stress} \end{array}
ight.$

- Equivalence of the strain energy release rate and SIF approach
- Mixed mode: G is scalar => mode contributions are additive
- Assumption: self-similar crack growth!!!

Self-similar crack growth: planar crack remains planar (da same direction ${\rm as}a$)
SIF in terms of compliance



A series of specimens with different crack lengths: measure the compliance C for each specimen -> dC/da -> K and G



Compliance-SIF $K = \sqrt{\sec \frac{\pi a}{W}} \sigma \sqrt{\pi a}$ $G = \frac{P^2}{2} \frac{dC}{dA} = \frac{P^2}{4R} \frac{dC}{da}$ $G = \frac{K^2}{E}$ $C = \int_{0}^{\infty} \frac{4}{EBW^2} \pi a \sec \frac{\pi a}{W} da + C_0$ $C_0 = \frac{\delta}{P} = \frac{H}{EBW}$ $\frac{P^2}{M} \frac{dC}{dC} = \frac{\sec \frac{\pi a}{W} \sigma^2 \pi a}{2\pi a}$ $\overline{4B} \ \overline{da}$ E $\frac{\pi a}{W} \sec\left(\frac{\pi a}{W}\right) = \tan\left(\frac{\pi a}{W}\right) \frac{\overline{W}^a}{\sin\left(\frac{\pi a}{W}\right)} \approx \tan\left(\frac{\pi a}{W}\right)$ $\frac{dC}{da} = \frac{\sec \frac{\pi a}{W} \sigma^2 \pi a 4B}{P^2 E}$ $\frac{dC}{d}$ $C = -\frac{4}{EB\pi} \ln\left(\cos\frac{\pi a}{W}\right) + \frac{H}{EBW}$ $\frac{1}{2}\pi a \sec \frac{\pi a}{W}$ da



$$C/C_0 = -\frac{4}{\pi} \frac{W}{H} \ln\left(\cos\frac{\pi a}{W}\right) + 1$$

compliance rapidly increases

K as a failure criterion

Failure criterion

$$K = K_c$$

$$f(a/W)\sigma\sqrt{\pi a} = K_c$$

fracture toughness

Problem 1: given crack length a, compute the maximum allowable applied stress

$$\sigma_{\max} = \frac{K_c}{f(a/W)\sqrt{\pi a}}$$

Problem 2: for a specific applied stress, compute the maximum permissible crack length (critical crack length)

$$f(a_c/W)\sigma\sqrt{\pi a_c} = K_c \rightarrow a_c$$

Problem 3: compute K covided crack length and stress at fracture

$$K_c = f(a_c/W)\sigma\sqrt{\pi a_c}$$

Example

A cylindrical pressure vessel with closed ends has a radius R = 1 m and thickness t = 40 mm and is subjected to internal pressure p. The vessel must be designed safely against failure by yielding (according to the von Mises yield criterion) and fracture. Three steels with the following values of yield stress σ_v and fracture toughness K_{Ic} are available for constructing the vessel.

| Steel | σ_{γ} (MPa) | $K_{Ic}(MPa\sqrt{m})$ |
|-----------------|-------------------------|-----------------------|
| A: 4340 | 860 | 100 |
| <i>B</i> : 4335 | 1300 | 70 |
| C: 350 Maraging | 1550 | 55 |

Fracture of the vessel is caused by a long axial surface crack of depth a. The vessel should be designed with a factor of safety S = 2 against yielding and fracture. For each steel:

- Plot the maximum permissible pressure p_c versus crack depth a_c ; (a)
- Calculate the maximum permissible crack depth a_c for an operating pressure (b) p = 12 MPa;
- Calculate the failure pressure p_c for a minimum detectable crack depth a =(c) 1 mm.

- 5. Elastoplastic fracture mechanics
 - 5.1 Introduction to plasticity
 - 5.2. Plastic zone models
 - 5.3. J Integral
 - 5.4. Crack tip opening displacement (CTOD)

5.2. Plastic zone models

- 1D Models: Irwin, Dugdale, and Barenbolt models
- 2D models:
 - Plastic zone shape
 - Plane strain vs. plane stress



Introduction

Griffith's theory provides excellent agreement with experimental data for brittle materials such as glass. For ductile materials such as steel, the surface energy (γ) predicted by Griffith's theory is usually unrealistically high. A group working under <u>G. R. Irwin</u> at the <u>U.S. Naval Research</u> Laboratory (NRL) during World War II realized that plasticity must play a significant role in the fracture of ductile materials.



(SSY) **Small-scale yielding:** LEFM still applies with minor modifications done by G. R. Irwin



Validity of K in presence of a plastic zone



Fracture process usually occurs in the inelastic region not the Kdominant zone.

is SIF a valid failure criterion for materials that exhibit inelastic deformation at the tip

Validity of K in presence of a plastic zone



Paradox of a sharp crack

At crack tip:

$$r = 0 \rightarrow \sigma_{ij} = \infty$$

An infinitely sharp crack is merely a mathematical abstraction.

Crack tip stresses are **finite** because (i) crack tip radius is finite (materials are made of atoms) and (ii) plastic deformation makes the crack blunt.



5.2.1 Plastic zone shape: 1D models

- 1st order approximation
- 2nd order Irwin model
- Strip yield models (Dugdale, and Barenbolt models)
- Effective crack length

Plastic correction: 1st order approximation

- A cracked body in a plane stress condition
- Material: elastic perfectly plastic with yield stress σ_{ys}



first order **approximation** of plastic zone size: equilibrium is not satisfied 175

2.Irwin's plastic correction



(b) Real crack with plastic zone

2.Irwin's plastic correction



$$r_p = rac{1}{3\pi} rac{K_I^2}{\sigma_{y_-}^2}$$

Plane strain

plastic zone: a CIRCLE !!!

Von Mises Yield Criterion $\sigma_e = \frac{1}{\sqrt{2}} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 \right]^{1/2} = \sigma_y$

Plane strain Plane stress $\sigma_1 = \sigma_2 = \sigma_{yy}$ $\sigma_1 = \sigma_2 = \sigma_{yy}, \ \sigma_3 = 0$ $\sigma_3 = \nu(\sigma_{xx} + \sigma_{yy}) = 2\nu\sigma_{yy}$ $\sigma_e = \sigma_1 = \sigma_y \quad \Rightarrow$ $r_{y} = \frac{1}{2\pi} \left(\frac{K_{I}}{\sigma_{ys}} \right)^{2} \quad \sigma_{e} = (1 - 2\nu)\sigma_{1} = \sigma_{y} \quad \Rightarrow$ $\sigma_{ys} = \frac{\sigma_{y}}{1 - 2\nu} \quad \Rightarrow$ $\sigma_{ys} = \sigma_y \quad \Rightarrow \quad$ $r_p = \frac{K_I^2}{\pi \sigma_u^2}$ $r_p = \frac{K_I^2}{\pi \sigma_{ys}^2} = (1-2\nu)^2 \frac{K_I^2}{\pi \sigma_y^2} \quad \Rightarrow \quad$ $r_p \approx \frac{K_I^2}{3\pi\sigma^2}$ (1 – 2 x 0.2)² = 0.36 X,r 178 X,r

3. Strip Yield Model

proposed by Dugdale and Barrenblatt

- Infinite plate with though thickness crack 2a
- Plane stress condition
- Elastic perfectly plastic material

Hypotheses:

- All <u>plastic deformation concentrates in</u> <u>a line in front of the crack</u>.
- The crack has an effective length which exceeds that of the physical crack by the length of the plastic zone.
- ho: chosen such that stress singularity at the tip disappears.



SIF for plate with normal force at crack





3. Strip Yield Model: Dugdale vs Barenblatt model

Dugdale: Uniform stress



More appropriate for polymers

Barenblatt: Linear stress



More appropriate for <u>metals</u>

5.2.1 Plastic zone shape: 1D models

- 1st order approximation
- 2nd order Irwin model
- Strip yield models (Dugdale, and Barenbolt models)
- Effective crack length



 $K_{I_{\text{eff}}} = f(a_{\text{eff}})$, (for example $K_{I_{\text{eff}}} = \sigma_0 \sqrt{\pi a_{\text{eff}}}$ infinite domain) \Rightarrow Nonlinear equation to solve for $K_{I_{\text{eff}}}$ and a_{eff}

Example

Consider an infinite plate with a central crack of length 2a subjected to a uniaxial stress perpendicular to the crack plane. Using the Irwin's model for a plane stress case, show that the effective SIF is given as follows

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$$K_{\text{eff}} = \frac{\sigma \sqrt{\pi a}}{\left[1 - 0.5 \left(\frac{\sigma}{\sigma_{ys}}\right)^2\right]^{1/2}}$$

Solution:

The effective crack length is $a+r_1$ The effective SIF is thus $K_{
m eff}=\sigma$

with

$$K_{\text{eff}} = \sigma \sqrt{\pi \left(a + r_p\right)}$$
$$r_p = \frac{1}{2\pi} \left(\frac{K_{I_{\text{eff}}}}{\sigma_{ys}}\right)^2$$

Consider a large central cracked plate subjected to a uniform stress of 130 MPa. The fracture toughness $K_c=50MPaVm$, the yield strength $\sigma_{ys}=420MPa$.

(a) What is the maximum allowable crack length?(b) What is the maximum crack length if plastic correction is taken into account. Plane stress and Irwin's correction.

Solution:

(a)
$$2a = 94.2 \text{ mm}$$

(b) $K_{\text{eff}} = \frac{\sigma \sqrt{\pi a}}{\left[1 - 0.5 \left(\frac{\sigma}{\sigma_{ys}}\right)^2\right]^{1/2}}$ (previous slide)
 $K_{\text{eff}} = K_c \rightarrow 2a = 89.7 \text{ mm}$

Effective crack length



Validity of LEFM solution



LEFM is better applicable to materials of high yield strength and low fracture toughness

5.2.2 Plastic zone shape: 2D models

- 2D models
- plane stress versus plane strain plastic zones

Plastic yield criteria







Plastic zone shape

von-Mises criterion

$$\sigma_e = \sigma_{ys}$$

 $\sigma_{e} = \frac{1}{\sqrt{2}} \left[(\sigma_{1} - \sigma_{2})^{2} + (\sigma_{1} - \sigma_{3})^{2} + (\sigma_{2} - \sigma_{3})^{2} \right]^{1/2}$ $\sigma_{1}, \sigma_{2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \left[\left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right)^{2} + \tau_{xy^{2}} \right]^{1/2}$

Principal stresses:

Mode I, principal stresses

$$\sigma_{1} = \frac{K_{I}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2}\right)$$

$$\sigma_{2} = \frac{K_{I}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2}\right)$$

$$\sigma_{3} = \begin{cases} 0 & \text{plane stress} \\ \frac{2\nu K_{I}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} & \text{plane strain} \end{cases}$$

$$r_{y}(\theta) = \frac{1}{4\pi} \left(\frac{K_{I}}{\sigma_{ys}}\right)^{2} \left[1 + \cos \theta + \frac{3}{2} \sin^{2} \theta\right] \qquad \text{plane stress}$$

$$r_{y}(\theta) = \frac{1}{4\pi} \left(\frac{K_{I}}{\sigma_{ys}}\right)^{2} \left[(1 - 2\mu)^{2}(1 + \cos \theta) + \frac{3}{2} \sin^{2} \theta\right] \text{plane strain}$$

Plastic zone shape

von-Mises criterion

Tresca criterion

$$\left(r_y(\theta) = \frac{1}{4\pi} \left(\frac{K_I}{\sigma_{ys}}\right)^2 \left[1 + \cos\theta + \frac{3}{2}\sin^2\theta\right]\right)$$

plane stress





Plastic zone shape: Mode I-III



Plastic zone sizes:Summary

| criterion | state | r_y or r_p | $\frac{r_y r_p }{(K_I / \sigma_y)^2}$ |
|-----------|---|--|--|
| Von Mises | plane stress | $\frac{1}{2\pi} \left(\frac{K_I}{\sigma_y}\right)^2$ | 0.1592 |
| Von Mises | plane strain | $\frac{1}{18\pi} \left(\frac{K_I}{\sigma_y}\right)^2$ | 0.0177 |
| Tresca | plane stress | $\frac{1}{2\pi} \left(\frac{K_I}{\sigma_y}\right)^2$ | 0.1592 |
| Tresca | plane strain $\sigma_1 > \sigma_2 > \sigma_3$ | $\frac{1}{18\pi} \left(\frac{K_I}{\sigma_y}\right)^2$ | 0.0177 |
| Tresca | plane strain $\sigma_1 > \sigma_3 > \sigma_2$ | 0 | 0 |
| Irwin | plane stress | $\frac{1}{\pi} \left(\frac{K_I}{\sigma_y} \right)^2$ | 0.3183 |
| Irwin | plane strain (pcf = 3) | $\frac{1}{\pi} \left(\frac{K_I}{3\sigma_y} \right)^2$ | 0.0354 |
| Dugdale | plane stress | $\frac{\pi}{8} \left(\frac{K_I}{\sigma_y}\right)^2$ | 0.3927 |
| Dugdale | plane strain (pcf = 3) | $\frac{\pi}{8} \left(\frac{K_I}{3\sigma_y}\right)^2$ | 0.0436 |

Source: Schreurs (2012)

What is the problem with these 2D shape estimates?

Stress not redistributed

stress redistribution (approximately) in 1D



Stress redistributed for 2D

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Dodds, 1991, FEM solutions Ramberg-Osgood material model

$$\frac{\varepsilon}{\varepsilon_o} = \frac{\sigma}{\sigma_o} + \alpha \left(\frac{\sigma}{\sigma_o}\right)^n$$



Effect of definition of yield (some level of ambiguity)

- Low n: High strain-hardening.
- $n \to \infty$: Similar to elastic perfectly plastic.



Effect of strain-hardening: Higher hardening (lower n) => smaller zone
Plane stress/plane strain



Plane stress/plane strain



Higher percentage of plate thickness is in plane strain mode for thicker plates



Plane stress/plane strain: Fracture loci



Loci of maximum shear stress for plane stress and strain

Plane stress/plane strain: What thicknesses are plane stress?

• As $\frac{\left(\frac{K}{\sigma_{ys}}\right)^2}{B}$ increases:

- 1. The plastic zone expands (load is increasing)
- 2. Plastic zone transitions from plane strain to plain stress



Plane stress/plane strain Toughness vs. thickness



Fracture toughness tests

- Prediction of failure in real-world applications: need the value of fracture toughness
- Tests on cracked samples: PLANE STRAIN condition!!!

Compact Tension

$$K_{I} = \frac{P}{B\sqrt{W}} \frac{\left(2 + \frac{a}{W}\right) \left[0.886 + 4.64\frac{a}{W} - 13.32\left(\frac{a}{W}\right)^{2} + 14.72\left(\left(\frac{a}{W}\right)^{3} - 5.6\left(\frac{a}{W}\right)^{4}\right]}{\left(1 - \frac{a}{W}\right)^{3/2}}$$



ASTM (based on Irwin's model) for plane strain condition (σ_Y is the yield stress NOT the adjusted $\sigma_{ys} \approx \sigma_Y/(1 - 2\nu)$):

$$a, B, (W-a) \ge 2.5 \left(\frac{K_{L}}{\sigma_{Y}}\right)^2$$

Fracture toughness test



Linear fracture mechanics is only useful when the plastic zone size is much smaller than the crack size

5.3. J Integral (Rice 1958)

Introduction

Idea

Replace complicated plastic model with nonlinear elasticity (no unloading)



J-integral

Eshelby, Cherepanov, 1967, Rice, 1968

Components of J integral vector

$$J_k = \int_{\Gamma} \left(W n_k - t_i \frac{\partial u_i}{\partial x_k} \right) \, d\Gamma$$

• J integral in fracture

$$J = J_1 = \int_{\Gamma} \left(W n_1 - t_i \frac{\partial u_i}{\partial x_1} \right) \, d\Gamma \, \left[\frac{\mathsf{N}}{\mathsf{m}} \right]$$

• strain energy density

$$W = \int_0^{\mathcal{E}_{pq}} \sigma_{ij} \, d\varepsilon_{ij}$$

Surface traction

$$t_i = \sigma_{ij} n_j$$



J-integral

Wikipedia

Rice used J1 for fracture characterization

$$egin{aligned} J &= \int_{\Gamma} \left(W n_1 - t_i rac{\partial u_i}{\partial x_1}
ight) d\Gamma \ J &= \int_{\Gamma} \left(W dx_2 - t_i rac{\partial u_i}{\partial x_1} ds
ight) \ \left[rac{N}{m}
ight] \end{aligned}$$



J-integral

(1) J=0 for a closed path

(2) is path-independent



5.3. 1. Path independence of J

J Integral zero for a closed loop



$$J_k = \int_{\Gamma} \left(\frac{\partial W}{\partial x_j} \delta_{jk} - \sigma_{ij,j} u_{i,k} - \sigma_{ij} u_{i,kj} \right) \mathrm{d}\Omega$$

and using chain rule:
$$\frac{\partial W}{\partial x_j} = \frac{\partial W}{\partial \epsilon_{mn}} \frac{\partial \epsilon_{mn}}{\partial x_j} \qquad \longrightarrow$$

$$J_k = \int_{\Omega} \left(\frac{dW}{d\varepsilon_{mn}} \frac{\partial \varepsilon_{mn}}{\partial x_j} \,\delta_{jk} - \sigma_{ij,j} u_{i,k} - \sigma_{ij} u_{i,kj} \right) \, d\Omega$$

J Integral zero for a closed loop

$$J_{k} = \int_{\Omega} \left(\frac{dW}{d\varepsilon_{mn}} \frac{\partial \varepsilon_{mn}}{\partial x_{j}} \delta_{jk} - \sigma_{ij,j} u_{i,k} - \sigma_{ij} u_{i,kj} \right) d\Omega$$

$$I_{k} = \int_{\Omega} \left(\frac{dW}{d\varepsilon_{mn}} \frac{\partial \varepsilon_{mn}}{\partial x_{j}} \delta_{jk} - \sigma_{ij,j} u_{i,k} - \sigma_{ij} u_{i,kj} \right) d\Omega$$

$$\sigma_{mn} = \frac{\partial W}{\partial \varepsilon_{mn}}$$

$$\sigma_{mn} = \frac{\partial W}{\partial \varepsilon_{mn}}$$

$$\varepsilon_{mn} = \frac{1}{2} (u_{m,n} + u_{n,m})$$

$$\sigma_{ij,j} = 0$$

$$J_{k} = \int_{\Omega} \left\{ \frac{1}{2} \sigma_{mn} (u_{m,nk} + u_{n,mk}) - \sigma_{ij} u_{i,kj} \right\} d\Omega =$$
$$\int_{\Omega} \left(\sigma_{mn} u_{m,nk} - \sigma_{ij} u_{i,kj} \right) d\Omega = 0$$

symmetry $\sigma_{mn} = \sigma_{nm}$

Path independence of J-integral

J is zero over a closed path

$$0 = J_{ABCDA} = J_{AB} + J_{BC} + J_{CD} + J_{DA}$$

$$J = \int_{\Gamma} \left(W dx_2 - t_i \frac{\partial u_i}{\partial x_1} ds \right)$$
AB, CD: traction-free crack faces

 $t_i = 0, dx_2 = 0$ (crack faces: parallel to x-axis)



5.3. 2. Relation between J and G (energy release property of J)

Energy release rate of J integral: Assumptions

- 1. Homogeneous body
- 2. Linear or non-linear elastic solid
- 3. No inertia, or body forces; no initial stresses
- 4. No thermal loading
- 5. 2-D stress and deformation field
- 6. Plane stress or plane strain
- 7. Mode I loading
- 8. Stress free crack

Energy release rate of J-integral



$$J-integral \qquad \mathbf{A}: \mathbf{B} = 0$$

$$\frac{\partial W}{\partial a} = \sigma_{ij} \frac{1}{2} \frac{\partial}{\partial a} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$\frac{\partial W}{\partial a} = \sigma_{ij} \frac{\partial}{\partial a} \frac{\partial u_i}{\partial x_j} = \sigma_{ij} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial a}$$

$$\int_{A'} \sigma_{ij} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial a} dA = \int_{\Gamma} \sigma_{ij} n_j \frac{\partial u_i}{\partial a} ds$$

$$\frac{\int_{A'} \frac{\partial W}{\partial a} dA = \int_{\Gamma} t_i \frac{\partial u_i}{\partial a} ds}{\sigma_{ij,j} = 0}$$
Gauss theorem
$$\frac{d\Pi}{da} = -\int_{A} \underbrace{\left(\frac{\partial W}{\partial x} \right)}_{J} dA + \int_{\Gamma} t_i \frac{\partial u_i}{\partial x} ds$$
Gauss theorem, $n_x ds = dy$
J-integral is equivalent to the energy release rate for a nonlinear elastic material under quasi-static condition.

Generalization of J integral

- Dynamic loading
- Surface tractions on crack surfaces
- Body force
- Initial strains (e.g. thermal loading)
- Initial stress from pore pressures

cf. Saouma 13.11 & 13.12 for details

5.3. 3. Relation between J and K

J-K relationship

$$G=\frac{K_I^2}{E'}+\frac{K_{II}^2}{E'}+\frac{K_{III}^2}{2\mu}$$

By idealizing plastic deformation as nonlinear elastic, Rice was able to generalize the energy release rate to nonlinear materials.

$$-\frac{d\Pi}{da} = \int_{\Gamma} \left(W dy - t_i \frac{\partial u_i}{\partial x} ds \right)$$

(previous slide)

J-integral: very useful in numerical computation of SIFs

J-K relationship

In fact both J1 (J) and J2 are related to SIFs:

$$J = J_1 - iJ_2 \quad \text{Hellen and Blackburn (1975)}$$
$$= \frac{(1+\nu)(1+\kappa)}{4E} (K_I^2 + K_{II}^2 + 2iK_IK_{II})$$

$$J_1 = \frac{K_I^2 + K_{II}^2}{E'}$$
$$J_2 = \frac{-2K_IK_{II}}{E'}$$

$$E' = \begin{cases} E & \text{plane strain} \\ \frac{E}{1-\nu^2} & \text{plane stress} \end{cases}$$

Note that if $K_I = a, K_{II} = b$ is a solution the general solution is: $K_I = \pm a, K_{II} = \pm b$ and $K_I = \pm b, K_{II} = \pm a$

5.3. 4. Energy Release Rate, crack growth and R curves

Goal: Obtain J from P- Δ Curve

- $J=-\frac{\partial \Pi}{\partial A}=-\frac{1}{B}\frac{\partial \Pi}{\partial a}$
- $\Pi = U_e W$: Potential energy
- W: External work
- $U_e = \int_V e \, \mathrm{dv}$: Internal energy
- $e(\epsilon_0)$ (or $w(\epsilon_0)$) = $\int_0^{\epsilon_0} \sigma(\epsilon) d\epsilon$
- <u>1. Load Control</u> (P fixed, u increases):

$$W = Pu \quad \Rightarrow \quad \Pi = U_e - Pu$$

$$\begin{split} \Pi &= -U_e^* \quad \text{load control} \\ U_e^* &= \int_0^P u \mathrm{d}P \text{ (complimentary strain energy)} \end{split}$$

$$J = \frac{1}{B} \left(\frac{\partial U_e^*}{\partial a} \right)_P \text{ (load control)}$$

Note $U_e = U_e^* = \frac{1}{2}Pu$ for linear solid



2. Displacement Control (u fixed, P decreases):

$$W = 0 \quad \Rightarrow \quad \Pi = U_e$$

$$\Pi = U_e$$
 displacement control

$$J = -\frac{1}{B} \left(\frac{\partial U_e}{\partial a} \right)_u \text{ (displacement control)}$$

Goal: Obtain J from P- Δ Curve



Goal: Analytical equation for J

$$J = -\lim_{a \to 0} \frac{\Delta \Pi}{B \Delta a}$$

= $-\lim_{a \to 0} \frac{U_{e2} - U_{e1} - W_{12}}{B \Delta a}$
= $-\lim_{a \to 0} \frac{1}{B} \left\{ \frac{U_{e2} - U_{e1}}{a_2 - a_1} - \frac{P_1 + P_2}{2} \frac{u_2 - u_1}{a_2 - a_1} \right\}$
= $\frac{1}{B} \left\{ -\frac{\mathrm{d}U_e}{\mathrm{d}a} + P \frac{\mathrm{d}u}{\mathrm{d}a} \right\}$

$$P_1$$

$$P_2$$

$$P_2$$

$$A_2$$

$$A_4$$

$$A_2$$

$$A_4$$

$$A_1$$

$$A_3$$

$$u$$

$$u_1$$

$$u_2$$

$$J = \frac{1}{B} \left\{ -\frac{\mathrm{d}U_e}{\mathrm{d}a} + P\frac{\mathrm{d}u}{\mathrm{d}a} \right\} = \frac{1}{B} \left\{ -\frac{\mathrm{d}U_e}{\mathrm{d}u} + P \right\} \frac{\mathrm{d}u}{\mathrm{d}a}$$

- Compare this equation with $G = -\frac{u^2}{2B}\frac{dK}{da} = \frac{P^2}{2B}\frac{dC}{da}$
- For linear case $U_e = \frac{1}{2}Ku^2$ and P = Ku. Show that equation for J agrees for the two for G

Goal: Experimental evaluation of J

- P-∆ curves for different crack lengths a →
 J as a function of ∆
- Rice proposes a method to obtain J with <u>only one</u> <u>test</u> for certain geometries

cf. Anderson 3.2.5 for details

Landes and Begley, ASTM 1972



Crack growth resistance curve

- A: R curve is nearly vertical:
 - small amount of apparent crack growth from blunting
- J_{I_c} measure of ductile fracture toughness
- Tearing modulus $T_R = \frac{E}{\sigma_0^2} \frac{dJ_R}{da}$ is a measure of crack <u>stability</u>





Crack growth and stability



Crack Size

- The J_R and J are similar to R and G curves for LEFM:
 - Crack growth can happen when $J = J_R$
 - Crack growth is unstable when

$$\frac{\mathrm{d}J}{\mathrm{d}a} > \frac{\mathrm{d}J_R}{\mathrm{d}a}$$

5.3.7. Fracture mechanics versus material (plastic strength

Governing fracture mechanism and fracture toughness



Fracture vs. Plastic collapse



Example

<u>Example 4.11</u> Estimate the failure load under uniaxial tension for a centre-cracked panel of aluminium alloy of width *W*=500 mm, and thickness *B*=4 mm, for the following values of crack length 2a = 20 mm and 2a = 100 mm. Yield stress $\sigma_y = 350$ MPa and fracture toughness $K_{lc}=70$ MPa \sqrt{m}

<u>Solution</u> There are two possible failure modes: plastic collapse and brittle fracture. We will assess the load level required for each mode to prevail.

(i) 2a = 20 mm.

Plastic collapse load $F_{pc} = \sigma_{ys} \cdot (W - 2a) \cdot B = 672$ kN

Fracture load
$$F_c = \sigma_c \cdot W \cdot B$$
 where $\sigma_c = \frac{K_{IC}}{\sqrt{\pi a \sec(\pi a / W)}} = 394.6$ MPa

thus $F_{\rm c}$ = 790 kN.

The actual failure load is the smaller of the above results, 672 kN.

(ii) 2a = 100 mm.

Plastic collapse load $F_{pc} = \sigma_{ys} \cdot (W - 2a) \cdot B = 560 \text{ kN}$

Fracture load
$$F_c = \sigma_c \cdot W \cdot B$$
 where $\sigma_c = \frac{K_{IC}}{\sqrt{\pi a \sec(\pi a / W)}} = 172.2$ MPa

thus $F_c = 334.57$ kN.

The actual failure load is the smaller of the above results, 334.6 kN.

5.3. 5. Plastic crack tip fields; Hutchinson, Rice and Rosengren (HRR) solution

Ramberg–Osgood model





- Elastic model:
 Unlike plasticity unloading in on the same line
- Higher n closer to elastic perfectly plastic
Hutchinson, Rice and Rosengren(HRR) solution

• Near crack tip "plastic" strains dominate:

$$\frac{\epsilon}{\epsilon_0} = \alpha \left(\frac{\sigma}{\sigma_0}\right)^n \qquad \bigstar$$

- Assume the following r dependence for σ and ϵ

$$\sigma = \frac{c_1}{r^x}$$
$$\varepsilon = \frac{c_2}{r^y}$$

1. Bounded energy:

$$\sigma \varepsilon \propto \frac{1}{r} \Rightarrow x + y = 1$$

$$x = \frac{1}{1 + n}$$

$$y = \frac{n}{1 + n}$$

$$y = nx$$

HRR solution: Local stress field based on J

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• Final form of HRR solution:

$$\sigma_{ij} = \sigma_0 \left(\frac{EJ}{\alpha \sigma_0^2 I_n r}\right)^{\frac{1}{n+1}} \tilde{\sigma}_{ij}(n,\theta)$$
$$\epsilon_{ij} = \frac{\alpha \sigma_0}{E} \left(\frac{EJ}{\alpha \sigma_0^2 I_n r}\right)^{\frac{n}{n+1}} \tilde{\epsilon}_{ij}(n,\theta)$$

J plays the role of K for local σ , ϵ , u fields



HRR solution



HRR solution: Angular functions







Stress is still singular but with a weaker power of singularity!

5.3. 6. Small scale yielding (SSY) versus large scale yielding (LSY)

Limitations of HRR solution

Limitations of HRR analysis

Plastic Zone

- Small strain: $\epsilon = \frac{1}{2} \left(\nabla \mathbf{u} + \nabla^{\mathrm{T}} \mathbf{u} \right)$ (accurate for $\epsilon \leq 0.1$)
- Small deformation theory (*e.g.*, not using PK stresses, etc)
- Elastic HRR model instead of plastic model





 $\delta: \text{ Crack tip opening}$ $\delta \propto \frac{K^2}{\sigma_y E}$ $\delta \propto \frac{K^2}{\sigma_y E}$ $\delta \propto \frac{K^2}{\sigma_y E}$ Blunted crack

ASTM 1979

From SSY to LSY



From SSY to LSY



LSY: When a single parameter (G, K, J, CTOD) is not enough?

- Under considerable plastic deformation and crack propagation when unloading and non-proportional zones grow out of J dominant zone with crack propagation. Reasons are:
 - Unloading: In J integral analysis plastic model was replaced by a nonlinear solid
 - <u>Single-parameter identification</u> not valid since various stress components increase at different rates



LSY: When a single parameter (G, K, J, CTOD) is not enough? <u>T stress</u>



LSY: When a single parameter (G, K, J, CTOD) is not enough? <u>J-Q theory</u>

- -Q parameter (J-Q theory) Valid for nonlinear analysis
 - * Added as a hydrostatic shift in front of crack to (HRR) stress fields

$$\sigma_{ij} \approx (\sigma_{ij})_{T=0} + Q\sigma_0 \delta_{ij} \quad \left(|\theta| \le \frac{\pi}{2}\right)$$

* Similar to T positive Q increases triaxiality and reduces fracture resistance

$$J_c = J_c(Q)$$

- More number of parameters: With extensive deformation two-parameter models such as K, T or J, Q evnetully break. $_{300}$







n: strain hardening in HRR analysis

5.4. Crack tip opening displacement (CTOD), relations with J and G

Crack Tip Opening Displacement





У

COD

$$v = \frac{\kappa + 1}{4\mu} \sigma \sqrt{a^2 - x^2}$$

COD is zero at the crack tips.

Crack Tip Opening Displacement: First order aproximation





COD is taken as the separation of the faces of the effective crack at the tip of the physical crack



(Irwin's plastic correction, **plane stress**)

Crack Tip Opening Displacement: Strip yield model



$$u_{y} = \frac{2}{E} \operatorname{Im} \tilde{Z} = \frac{4\sigma_{YS}}{\pi E} \left[a \operatorname{coth}^{-1} \left(\frac{1}{a_{1}} \sqrt{\frac{a_{1}^{2} - z^{2}}{1 - k^{2}}} \right) - z \operatorname{coth}^{-1} \left(\frac{k}{z} \sqrt{\frac{a_{1}^{2} - z^{2}}{1 - k^{2}}} \right) \right]$$

$$\mathcal{Z} = \mathbf{a} \implies \delta = 2u_{y} = \frac{8\sigma_{YS}a}{\pi E} \ln\left(\frac{1}{k}\right) = \frac{8\sigma_{YS}a}{\pi E} \left[\frac{1}{2}\left(\frac{\pi}{2}\frac{\sigma}{\sigma_{YS}}\right)^{2} + \frac{1}{12}\left(\frac{\pi}{2}\frac{\sigma}{\sigma_{YS}}\right)^{4} + \cdots\right] \implies$$

For $\sigma/\sigma_{YS} \to 0$ $\delta = \frac{K_{I}^{2}}{\sigma_{YS}E}$

CTOD-G-K relation

Wells observed:

The degree of crack blunting increases in proportion to the toughness of the material

$$\delta = \frac{4}{\pi} \frac{K_I^2}{\sigma_{ys} E}$$

$$G_I = \frac{K_I^2}{E}$$

$$G_I = \frac{\pi}{4} \sigma_{ys} \delta$$

material property independent of specimen and crack length (confirmed by experiments)

 $\delta = \delta_c$

Fracture occurs

Under conditions of SSY, the fracture criteria based on the stress intensity factor, the strain energy release rate and the crack tip opening displacement are equivalent.

CTOD-J relation

• When SSY is satisfied G = J so we expect:

$$G = m\sigma_y \delta \quad \Rightarrow \quad J = m\sigma_y \delta$$

- In fact this equation is valid well beyond validity of LEFM and SSY
- E.g. for HRR solution Shih showed that:

 δ is obtained by 90 degree method:
 Deformed position corresponding to r* = r and φ = -π forms 45 degree w.r.t crack tip)

$$\frac{\delta}{2} = u_y(r^*, \pi) = r^* - u_x(r^*, \pi)$$

$$r^* = \left(\frac{\alpha \sigma_o}{E}\right)^{un} \{\tilde{u}_x(\pi, n) + \tilde{u}_y(\pi, n)\}^{\frac{n+1}{n}} \frac{J}{\sigma_o I_n} \implies J = m\sigma_0 \delta$$

for
$$m = \frac{1}{d_n}, d_n = \frac{2\tilde{u}_y(\pi, n) \left[\frac{\alpha \sigma_o}{E} \{\tilde{u}_x(\pi, n) + \tilde{u}_y(\pi, n)\}\right]^{1/n}}{I_n}$$



0

0

0.1

0.2

0.3

1/n

0.001

0.5

CTOD experimental determination



For high elastic deformation contribution, elastic corrections should be added

6. Computational fracture mechanics

6.1. Fracture mechanics in Finite Element Methods6.2. Traction Separation Relations (TSRs)

6.1Fracture mechanics in Finite Element Methods (FEM)

6.1.1. Introduction to Finite Element method

6.1.2. Singular stress finite elements

6.1.3. Extraction of K (SIF), G

6.1.4. J integral

- 6.1.5. Finite Element mesh design for fracture mechanics
- 6.1.6. Computational crack growth

6.1.7. Extended Finite Element Method (XFEM)

Numerical methods to solve PDEs

- Finite Difference (FD) & Finite Volume (FV) methods
- FEM (Finite Element Method)
- BEM (Boundary Element Method)
- MMs (Meshless/Meshfree methods)







x2

Fracture models

- Discrete crack models (discontinuous models):
 Cracks are explicitly modeled
 - LEFM
 - EPFM
 - Cohesive zone models
- Continuous models: Effect of (micro)cracks and voids are incorporated in bulk damage
 - Continuum damage models
 - Phase field models
- Peridynamic models: Material is modeled as a set of particles

Finite Element Method





 $h_3 = \frac{1}{4} \left(1 + x_1 \right) \left(1 + x_2 \right)$







Global level: The nodal dof at the center node is shared by all four elements

Element level: 4 Shape functions for a linear quad element



6.1Fracture mechanics in Finite Element Methods (FEM)

6.1.1. Introduction to Finite Element method

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6.1.7. Extended Finite Element Method (XFEM)

Isoparametric Elements



- Geometry is mapped from a parent element to the actual element
- The same interpolation is used for geometry mapping and FEM solution (in the figure 2nd order shape functions are used for solution and geometry)
- Geometry map and solution are expressed in terms of ξ

parent element

Actual element



Order of element

based on the order)

Singular crack tip solutions



• NLFM (PFM): For HRR solution stress $\frac{1}{r^{\frac{1}{n+1}}}$ and strain $\frac{1}{r^{\frac{n}{n+1}}}$ are still singular \Rightarrow - for elastic-perfectly plastic $(n \to \infty)$ stress is bounded and strain is $\frac{1}{r}$ singular

Isoparametric singular elements



Motivation: 1D quadrature element

Find α that yields ϵ singularity at x_1

Isoparametric element:

1. Geometry:

$$x = \Sigma_{i=1}^{3} N_{i}(\xi) x_{i} \implies x = L\left\{\xi^{2}(\frac{1}{2}-\alpha) + \frac{1}{2}\xi + \alpha\right\} \xrightarrow{\left(\begin{matrix} n \\ n \\ n \\ \frac{dx}{d\xi} = L\left\{2\xi(\frac{1}{2}-\alpha) + \frac{1}{2}\right\} \xrightarrow{\left(\begin{matrix} n \\ n \\ \frac{dx}{d\xi} = L\left\{2\xi(\frac{1}{2}-\alpha) + \frac{1}{2}\right\}\right\}} \xrightarrow{\left(\begin{matrix} n \\ \frac{dx}{d\xi} \\ \frac{dx}{d\xi} = L\left\{2\xi(\frac{1}{2}-\alpha) + \frac{1}{2}\right\} \xrightarrow{\left(\begin{matrix} n \\ \frac{dx}{d\xi} \\ \frac{dx}{d\xi} \\ \frac{dx}{d\xi} = -1\right) = 0 \xrightarrow{\left(\begin{matrix} n \\ \frac{dx}{d\xi} \\ \frac{dx}{d\xi} \\ \frac{dx}{d\xi} \\ \frac{dx}{d\xi} = -1\right) = 0 \xrightarrow{\left(\begin{matrix} n \\ \frac{dx}{d\xi} \\ \frac{$$

 $x_3 = \alpha L$

2

 $\overset{x_1}{\longrightarrow} \overset{x_1}{\longrightarrow} \overset{x_1}{\longrightarrow}$

Motivation: 1D quadrature element

- 2. FEM solution
 - Displacement

$$u = \sum_{i=1}^{3} N_i(\xi) u_i \implies$$

$$u = u_1 \left[\frac{\xi(\xi - 1)}{2} \right] + u_2 \left[\frac{\xi(\xi + 1)}{2} \right] + u_3 \left[1 - \xi^2 \right] \Rightarrow$$

$$u = u_1 + \frac{\sqrt{x}}{\sqrt{L}} \left(-3u_1 - u_2 + 4u_3 \right) + \frac{2x}{L} \left(u_1 + u_2 - 2u_3 \right)$$

which matches \sqrt{x} from asymptotic displacement solution.

• Strain

$$\begin{array}{l} \epsilon &= \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}\xi} / \frac{\mathrm{d}x}{\mathrm{d}\xi} \\ \frac{\mathrm{d}x}{\mathrm{d}\xi} &= L\frac{\xi+1}{2} = \sqrt{xL} \\ \frac{\mathrm{d}u}{\mathrm{d}\xi} &= u_1 \left[\frac{2\xi-1}{2} \right] + u_2 \left[\frac{2\xi+1}{2} \right] - 2u_3 \xi \end{array} \right\} \Rightarrow \\ \epsilon &= \frac{1}{\sqrt{xL}} \left(-\frac{3}{2}u_1 - \frac{1}{2}u_2 + 2u_3 \right) + \frac{1}{L} \left(2u_1 + 2u_2 - 4u_3 \right) \end{array}$$

Strain field too matches asymptotic term $\frac{1}{\sqrt{r}}$



Moving singular ϵ position

Strain singularity at ξ means

$$\frac{\mathrm{d}x}{\mathrm{d}\xi} = L\left\{2\xi(\frac{1}{2} - \alpha) + \frac{1}{2}\right\}$$

must be zero. Accordingly,

Singularity at $infinity(\xi \to -\infty)$

Singularity at crack $tip(\xi = -1)$

Singularity inside element(not of interest)($-1 < \xi < 0$)

 Transition elements: According to this analysis mid nodes of next layers move to ½ point from ¼ point

Lynn and Ingraffea 1977)



6.1Fracture mechanics in Finite Element Methods (FEM)

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6.1.7. Extended Finite Element Method (XFEM)

1. K from local fields



Mixed mode generalization:

1. K from local fields

2. Stress

$$K_I = \lim_{r \to 0} \left(\sqrt{2\pi r} \ \sigma_{22}|_{\theta=0} \right) \quad ; \quad K_{II} = \lim_{r \to 0} \left(\sqrt{2\pi r} \ \sigma_{12}|_{\theta=0} \right)$$



Stress based method is less accurate because:

- Stress is a derivative field and generally is one order less accurate than displacement
- Stress is singular as opposed to displacement
- Stress method is much more sensitive to where loads are applied (crack surface or far field)

2. K from energy approaches

- 1. Elementary crack advance (two FEM solutions for a and $a + \Delta a$)
- 2. Virtual Crack Extension: Stiffness derivative approach
- 3. J-integral based approaches (next section)

After obtaining G (or J=G for LEFM) K can be obtained from

$$K_I^2 = E'G$$
 $E' = \begin{cases} E & \text{plane stress} \\ \frac{E}{1-\nu^2} & \text{plane strain} \end{cases}$

2.1 Elementary crack advance

For fixed grip boundary condition perform two simulations (1, *a*) and (2, $a+\Delta a$): All FEM packages can compute strain (internal) energy U_i



Drawback:

- 1. Requires two solutions
- 2. Prone to Finite Difference (FD) errors

2.2 Virtual crack extension

• Potential energy is given by

$$\Pi = \frac{1}{2} \lfloor u \rfloor [K] \{u\} - \lfloor u \rfloor \{p\}$$

$$-G = \frac{\partial \Pi}{\partial a} = \frac{\partial \lfloor u \rfloor}{\partial a} [K] \{u\} + \frac{1}{2} \lfloor u \rfloor \frac{\partial [K]}{\partial a} \{u\} - \frac{\partial \lfloor u \rfloor}{\partial a} \{P\} - \lfloor u \rfloor \frac{\partial \{P\}}{\partial a}$$
$$= -\frac{\partial \lfloor u \rfloor}{\partial a} \underbrace{\left([K] \{u\} - \{P\}\right)}_{0} + \frac{1}{2} \lfloor u \rfloor \frac{\partial [K]}{\partial a} \{u\} - \lfloor u \rfloor \frac{\partial \{P\}}{\partial a}$$
$$G = -\frac{1}{2} \lfloor u \rfloor \frac{\partial [K]}{\partial a} \{u\} + \lfloor u \rfloor \frac{\partial \{P\}}{\partial a}$$
Furthermore when the loads are constant:
$$\mathcal{G} = \frac{K_I^2}{E'} = -\frac{1}{2} [\mathbf{u}]^T \frac{\partial [\mathbf{K}]}{\partial a} [\mathbf{u}]$$

- Only the few elements that are distrorted contribute to $\frac{\partial K}{\partial a}$
- We may not even need to form elements and assemble K for a and a + Δa to obtain <u>∂K</u>. We can explicitly obtain <u>∂k^e</u> for elements affected by crack growth by computing derivatives of actual geometry of the element to parent geometry.
 - This method is equivalent to J integral method (Park 1974)




2.2 Virtual crack extension: Mixed mode

• For LEFM energy release rates G_1 and G_2 are given by

•

$$J_{1} = G_{1} = \frac{K_{I}^{2} + K_{II}^{2}}{E'} + \frac{K_{III}^{2}}{2\mu}$$
$$J_{2} = G_{2} = \frac{-2K_{I}K_{II}}{E'}$$

• Using Virtual crack extension (or elementary crack advance) compute G_1 and G_2 for crack lengths $a, a + \Delta a$

$$J_{1} = G_{1} = \frac{K_{I}^{2} + K_{II}^{2}}{E'} + \frac{K_{III}^{2}}{2\mu}$$

$$J_{2} = G_{2} = \frac{-2K_{I}K_{II}}{E'}$$

$$K_{I} = \frac{s \pm \sqrt{s^{2} + \frac{8G_{2}}{\alpha}}}{4}$$
Note that there are two sets of solutions!
$$K_{II} = \frac{s \pm \sqrt{s^{2} + \frac{8G_{2}}{\alpha}}}{4}$$

$$s = 2\sqrt{\frac{G_1 - G_2}{\alpha}}$$
 and $\alpha = \frac{(1+\nu)(1+\kappa)}{E}$

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6.1Fracture mechanics in Finite Element Methods (FEM)

6.1.1. Introduction to Finite Element method

6.1.2. Singular stress finite elements

6.1.3. Extraction of K (SIF), G

6.1.4. J integral

6.1.5. Finite Element mesh design for fracture mechanics

6.1.6. Computational crack growth

6.1.7. Extended Finite Element Method (XFEM)

J integral

Uses of J integral:

1. LEFM: Can obtain K_{I} and K_{II} from J integrals (G = J for LEFM)

$$J_{1} = G_{1} = \frac{K_{I}^{2} + K_{II}^{2}}{E'} + \frac{K_{III}^{2}}{2\mu}$$
$$J_{2} = G_{2} = \frac{-2K_{I}K_{II}}{E'}$$

2. Still valid for nonlinear (NLFM) and plastic (PFM) fracture mechanics

Methods to evaluate J integral:

1. Contour integral:

$$J_{1} = \int_{\Gamma} \left(w dy - \mathbf{t} \frac{\partial \mathbf{u}}{\partial x} d\Gamma \right)$$
$$J_{2} = \int_{\Gamma} \left(-w dx - \mathbf{t} \frac{\partial \mathbf{u}}{\partial y} d\Gamma \right)$$

- 2. Equivalent (Energy) **domain** integral (EDI):
 - Gauss theorem: line/surface (2D/3D) integral
 - Much simpler to evaluate computationally
 - Easy to incorporate plasticity, crack surface tractions, thermal strains, *etc.*
 - Prevalent method for computing J-integral



surface/volume integral

J integral: 1.Contour integral

• Stresses are available and also more accurate at Gauss points



J integral: 2.Equivalent Domain Integral (EDI)



J integral: 2. EDI: Derivation

Divergence theorem: Line/Surface (2D/3D) integral

Surface/Volume Integral





Simplified Case:

(Nonlinear) elastic, no thermal strain, no body force, traction free crack surfaces

$$J = \int_{A^*} \left[\sigma_{ij} \frac{\partial u_j}{\partial x_1} - w \delta_{1i} \right] \frac{\partial q}{\partial x_i} dA$$

This is the same as deLorenzi's approach where finds a physical interpretation (virtual crack extension)

$$\mathcal{G} = \frac{1}{\Delta a} \int_{\mathcal{A}} \left(\sigma_{ij} \frac{\partial u_j}{\partial x_1} - w \delta_{i1} \right) \frac{\partial \Delta x_1}{\partial x_i} dA$$



J integral: 2. EDI FEM Aspects

- Since $J_0 \rightarrow \theta$ the inner J_0 collapses to the crack tip (CT)
- J₁ will be formed by element edges
- By using spider web (rozet) meshes any reasonable number of layers can be used to compute J:



- Spider web (rozet) mesh:
 - One layer of triangular elements (preferably singular, quadrature point elements)
 - Surrounded by quad elements



J integral: 2. EDI FEM Aspects

• Shape of decreasing function *q*:



Pyramid q function

Plateau q function

 Plateau q function useful when inner elements are not very accurate: e.g. when singular/quarter point elements are not used



J integral: Mixed mode loading

• For LEFM we can obtain K_1 , K_1 , K_2 , from J_1 , J_2 , G_2 .

$$\bar{J}_k \Delta L = \int_{V^*} \left(\sigma_{ij} \frac{\partial u_i}{\partial x_k} \frac{\partial q}{\partial x_j} - w \frac{\partial q}{\partial x_k} \right) dV$$
$$G_{III} = \int_{V^*} \left(\sigma_{3j} \frac{\partial u_3}{\partial x_1} \frac{\partial q}{\partial x_j} - w^{III} \frac{\partial q}{\partial x_1} \right) dV$$

$$K_{I} = \frac{1}{2}\sqrt{E^{*}} \left(\sqrt{(J_{1} - J_{2} - G_{3})} + \sqrt{(J_{1} + J_{2} - G_{3})} \right)$$

$$K_{II} = \frac{1}{2}\sqrt{E^{*}} \left(\sqrt{(J_{1} - J_{2} - G_{3})} - \sqrt{(J_{1} + J_{2} - G_{3})} \right)$$

$$K_{III} = \sqrt{2\mu G_{3}}$$

$$E^{*} = E \left[\frac{1}{1 - \nu^{2}} + \left(\frac{\nu}{1 + \nu} \right) \frac{\varepsilon_{33}}{\varepsilon_{11} + \varepsilon_{22}} \right]$$

$$E^{*} = \frac{E}{1 - \nu^{2}} \text{ for plane strain and } E^{*} = E \text{ for plane stress}$$

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Different elements/methods to compute K



- J integral EDI method is by far the most accurate method
- Interpolation of K from *u* is more accurate from σ: 1) higher convergence rate, 2) nonsingular field. Unlike σ it is almost insensitive to surface crack or far field loading
- Except the first contour (J integral) or very small r the choice of element has little effect

Different elements/methods to compute K: Effect of adaptivity on local field methods



Even element h-refinement cannot improve K values by much particularly for stress based method

1. Crack surface meshing:

- Nodes in general should be duplicated:
 - Modern FEM can easily handle duplicate nodes



• If not, small initial separation is initially introduced



 When large strain analysis is required, initial mesh has finite crack tip radius. The opening should be smaller than 5-10 times smaller than CTOD. Why?



2. Quarter point vs mid point elements / Collapsed elements around the crack tip

• For LEFM singular elements triangle quarter elements are better than normal tri/quad, and quarter point quad elements (collapsed or not)



- Perturbation of quarter point by *e* results in $\partial(ge^2)$ error in *K* (*g* = *h*/a)
- For elastic perfectly plastic material collapsed quad elements (1st / 2nd order) are recommended



3. Shape of the mesh around a crack tip

- a) <u>Around the crack tip</u> triangular singular elements are recommended (little effect for EDI J integral method)
- b) Use <u>quad elements</u> (2nd order or higher) <u>around the first contour</u>
- c) <u>Element size</u>: Enough number of elements should be used in region of interest: r_s , r_p , large strain zone, etc.
- d) Use of transition elements away from the crack tip although increases the accuracy has little effect



4. Method for computing *K*

- Energy methods such as J integral and G virtual crack extension (virtual stiffness derivative) are more reliable
 - J integral EDI is the most accurate and versatile method
 - Least sensitive method to accuracy of FEM solution at CT particularly if plateau q is used



- K based on local fields is the least accurate and most sensitive to CT solution accuracy.
 - Particularly stress based method is not recommended.
 - Singular/ quarter point elements are recommended for these methods especially when K is obtained at very small r

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What's wrong with FEM for crack problems

- Element edges must conform to the crack geometry: make such a mesh is time-consuming, especially for 3D problems.
- Remeshing as crack advances: difficult. Example:



Capturing/tracking cracks



Fixed mesh

Crack tracking

XFEM enriched elements

Crack/void capturing by bulk damage models

Brief overview in the next section

Brief overview in continuum damage models

Fixed meshes

- Nodal release method (typically done on fixed meshes)
 - Crack advances one element edge at a time by <u>releasing FEM nodes</u>
 - Crack path is restricted by discrete geometry



 Also for cohesive elements they can be used for both extrinsic and intrinsic schemes. For intrinsic ones, cohesive surfaces between all elements induces an artificial compliance (will be explained later)

Adaptive meshes

• Adaptive operations align element boundaries with crack direction

Element splitting:

newly generated elements

Smoother crack path by element splitting:

cracks split through and propagate between



Cracks generated by refinement options

 x_2

C

Element edges move to desired direction

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Finite Elements for singular crack tip solutions $\begin{bmatrix} u = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi} \cos \frac{\theta}{2}} \left(\kappa - 1 + 2 \sin^2 \frac{\theta}{2} + 2 \sin^2 \frac{\theta}{2}} \\ v = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi} \sin \frac{\theta}{2}} \left(\kappa + 1 - 2 \cos^2 \frac{\theta}{2} + 1 - 2 \cos^2 \frac{\theta}{2}} \right)$

- Direct incorporation of singular terms: $\sum_{k=1}^{4} f_k \bar{u}_{ik} + K_{II} \left(Q_{1i} \sum_{k=1}^{4} f_k \bar{Q}_{1ik} \right) + K_{II} \left(Q_{2i} \sum_{k=1}^{4} f_k \bar{Q}_{2ik} \right)$ e.g. enriched elements by Benzley (1974), shape functions are enriched by K_I, K_{II} singular terms $u' = \left(1 + \frac{2x}{L} - 3\sqrt{\frac{x}{L}} \right) u'_1 + \left(4\sqrt{\frac{x}{L}} - 4\frac{x}{L} \right) u'_2 + \left(\frac{2x}{L} - \sqrt{\frac{x}{L}} \right) u'_3$
 - XFEM method falls into this group (discussed later)
 More accurate

• Quarter point (LEFM) and Collapsed Can be easily used in FEM software half point (Elastic-perfectly plastic) elements: By appropriate positioning of isoparametric element nodes create strain singularities

Extended Finite Element Method (XFEM)

Belytschko et al 1999

$$\mathbf{u}^{h}(\mathbf{x}) = \left[\sum_{I \in \mathcal{S}} N_{I}(\mathbf{x})\mathbf{u}_{I}\right] + \left[\sum_{J \in \mathcal{S}^{c}} N_{J}(\mathbf{x})\Phi(\mathbf{x})\mathbf{a}_{J}\right]$$

standard part enrichment part

Partition of Unity (PUM)

enrichment function

$$\sum_{J} N_{J}(\mathbf{x}) = 1 \longrightarrow \sum_{J} N_{J}(\mathbf{x}) \Phi(\mathbf{x}) = \Phi(\mathbf{x})$$

 $\Phi(\mathbf{x})$ known characteristics of the problem (crack tip singularity, displaceme jump etc.) into the approximate space.

XFEM: enriched nodes



enriched nodes = nodes whose support is cut by the item to be enriched

enriched node I: standard degrees of freedoms (dofs) and additional dofs

XFEM for LEFM



$$\Phi_2: \Phi_2(x^+) \neq \Phi_2(x^-)$$



XFEM for LEFM (cont.)

Crack tip enrichment functions:

$$\begin{bmatrix} u = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left(\kappa - 1 + 2\sin^2 \frac{\theta}{2}\right) \\ v = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left(\kappa + 1 - 2\cos^2 \frac{\theta}{2}\right) \end{bmatrix}$$

$$\begin{bmatrix} r \cos \frac{\theta}{2} & \sin \theta \end{bmatrix}$$

 $[B_{\alpha}] = \left[\sqrt{r}\sin\frac{\theta}{2}, \sqrt{r}\cos\frac{\theta}{2}, \sqrt{r}\sin\frac{\theta}{2}\sin\theta, \sqrt{r}\cos\frac{\theta}{2}\sin\theta\right]$

Crack edge enrichment functions:



XFEM for cohesive cracks Wells, Sluys, 2001

$$\mathbf{u}^{h}(\mathbf{x}) = \sum_{I \in \mathcal{S}} N_{I}(\mathbf{x}) \mathbf{u}_{I} + \sum_{J \in \mathcal{S}^{c}} N_{J}(\mathbf{x}) H(\mathbf{x}) \mathbf{a}_{J}$$



No crack tip solution is known, no tip enrichment!!!

not enriched to ensure zero crack tip opening!!!

$$H(\mathbf{x}) = \begin{cases} +1 & \text{if } (\mathbf{x} - \mathbf{x}^*) \cdot \mathbf{n} \ge 0 \\ -1 & \text{otherwise} \end{cases}$$

XFEM: examples





CENAERO, M. Duflot

Northwestern Univ.



Meshfree methods



Bordas et al.





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Elastic-plastic fracture

Shaofan Li 2012

6.2. Traction Separation Relations (TSRs)

Cohesive models

Cohesive models remove stress singularity predicted by Linear Elastic Fracture Mechanics (LEFM)



Sample applications of cohesive models

fracture of polycrystalline material





delamination of composites



fragmentation



Microcracking and branching



Cohesive models

 Traction Separation Relation (TSR): Relation between traction (stress) and displacement jump

 $\sigma = \tilde{\sigma} f(\delta/\tilde{\delta})$

- Parameters of a cohesive model(Only 2 out of 3 are needed)
 - Stress (traction) scale $\tilde{\sigma}$: Maximum traction in TSR
 - Displacement scale δ : Separation corresponding to maximum traction (extrinsic models) or maximum nonzero traction
 - <u>Work of Separation</u> $\tilde{\phi}$: Area under $\sigma \delta$ curve is the work needed to complete debond a unit surface area. This can be associated with G_c in LEFM theory.
- Types of Cohesive models
 - Intrinsic cohesive models:
 - * It has and initial hardening $\sigma-\delta$ part
 - * σ starts from 0
 - * Can be inserted in FEM mesh from the start of the simulation (along certain lines or between all elements)
 - Extrinsic cohesive models:
 - * Generally has only softening $\sigma-\delta$ behavior.
 - * σ starts from maximum stress $(\tilde{\sigma})$
 - * Should be adaptively inserted between elements when traction between elements approach $\tilde{\sigma}$



Process zone (Cohesive zone)

- Process zone Important Points and lengths
 - Nominal crack tip $x_{\rm CT}$ Generally corresponds to the point with maximum traction ($\tilde{\sigma}$)
 - Nominal trailing edge of the process zone $x_{\rm T}$: The point where traction goes to zero (or if asymptotically goes to zero taken when stress is arbitrary small e.g., 0.01 or 0.001 sigma.
 - Nominal leading edge of the process zone $x_{\rm L}$: When general crack-like (e.g., highly nonlinear response) starts. Often, $x_{\rm L}$ is set to $x_{\rm CT}$.
 - Process zone size $\Lambda = |x_{\rm L} x_{\rm T}|$: Characteristic length scale corresponding to cohesive models(L)

Cohesive model shape

Can have important influence on the response of cohesive model



- The shape can be based on ductile/brittle response of TSR and can make it intrinsic or extrinsic
Shape of TSR

- Cohesive model shape
 - Can have important influence on the response of cohesive model
 - The shape can be based on ductile/brittle response of TSR and can make it intrinsic or extrinsic



Cohesive model Table

Source: Namas Chandra, Theoretical and Computational Aspects of Cohesive Zone Modeling, Department of Mechanical Engineering, FAMU-FSU College of Engineering, Florida State University

| Year and author | Proposed Model | Model parameters | Problem solved | Model constants | Comments |
|--------------------------------|---|--|--|--|--|
| 1959, 1962 Barenblatt, G.I. | y=b | $K = \int_{0}^{4+d} \frac{G_{1}(t)dt}{\sqrt{t}} = \sqrt{\frac{\pi E T}{1-v^{2}}} ductile)$ $T = T_{p} + T_{1} = \sqrt{\frac{\pi E T_{1}}{1-v^{2}}} brittle)$ $T_{p} = work of separation for brit. mat$ $T_{1} = work of plastic deformation$ | Brittle materials | | First to propose the concept |
| 1960 Dugdale | | $\frac{a}{a} = 2 \sin^2\left(\frac{\pi}{4}\frac{\tau}{Y}\right)$ For small value of T/Y $\frac{s}{1} = 1.23\left(\frac{T}{Y}\right)^2$ | Steel sheets | Plastic zone ranges from 0.042 to 0.448 (in) | Cohesive stress equated to yield stress of material |
| 1987 Needlem an | T Polynomial fit T + Linear fit u u, | φ,,, = wk: of Sep. δ = normalizing Par. σ = Max. Stress | Particle-matrix decohesion | $\delta = 10^{9}$ to 10^{3} m C ohesive Energy 1 to 10 J/m ² $\sigma_{mm} = 1000$ MPa | Phenomenological model. predict normal separation |
| 1989 Rice & Wang | σ Model based on atomic fit of the σ _{exax} δ | E = Init. Young's mod h = normalizing Par. $\sigma_{max} = Max. Stress$ $\alpha = constant(\frac{\epsilon, b}{a^3} = 2\gamma)$ | Solute segregation | | Ascending part is equated to E Normal separation. Ignores shear separation |
| 1990 Needleman | T _n + Polynom inl Exponential for Tn. Lineat fit for Ts Δ_n/δ | t φ _{sep} = wk. of Sep. δ = normalizing Par. σ _{max} = Max. Stress | Particle-matrix decohesion | δ = 10° to 10 ³ m | predict normal separation |
| 1990 Needlem an | Exponential fit for norm al Tn Trignom etric fit for shear Ts T _n u _n t _n T _n U _n T | φ = wk. of Nor. Sep. φ = wk. of Shr. Sep. δ ,δ = critical displacements. σ max = Max. Stress 318 | Decohesion of interface under hydrostatic tensiom | $\delta_{a} = \delta_{t} =$ $2x10^{10} \text{to} 2x10^{\circ} \text{ m}$ $J/\phi_{a} = 0.57 - 2.59$ $\sigma_{mm}/\sigma_{0} = 2,3$ | Periodic shear traction to model Pieriels shear stress due to slip |

Cohesive model Table

Source: Namas Chandra, Theoretical and Computational Aspects of Cohesive Zone Modeling, Department of Mechanical Engineering, FAMU-FSU College of Engineering, Florida State University

| Year and author | Proposed Model | Model parameters | Problem solved | model constants | Comments | | |
|-----------------------------------|--|--|---|--|---|--|--|
| 1992 Tvergaard & Hutchinson | $\hat{\sigma}$ | $\begin{split} & \Gamma_{\circ} = \text{wk. of Separation.} \\ & \delta_{n}^{\circ}, \delta_{+}^{\circ} = \text{critical displ.} \\ & \hat{\sigma} = \text{Peak nor. trn./int. face str.} \\ & \lambda_{1}, \lambda_{2} = \text{factors governing} \\ & \text{shape of sep} \end{split}$ | Crack growth in elasto-plastic material, peeling of adhesive joints | $\begin{split} & \Gamma_{\mu} / \Gamma_{\bullet} = 0 - 10. \\ & (\Gamma_{\mu} = \text{P1 wk.}), \delta_{\mu}^{a} / \delta_{\dagger}^{a} = 1 \\ & \hat{\sigma} / \sigma_{y} = 0 - 14 \\ & \lambda_{1}, \lambda_{2} = 0.15, 0.5 \\ & \sigma_{y} / E = 1 / 300 \end{split}$ | Claims shape of separation law are melatively unimportant | | |
| 1993 Xu & Needleman | Exponential fit for both Tn and Tt $T_n \phi_n$ $T_t \phi_1$ $\Delta_1 + \phi_1$ $\Delta_1 + \phi_1$ | $\phi_n = wk. of Nor. Sep.$ $\phi_t = wk. of Shr. Sep.$ $\delta_n, \delta_t = critical displacements$ $\sigma_{max} = Max. Stress$ | Partic le-matrix decohesion | δ _n ,δ _t = 2x10 ⁻¹⁹ to 2x10 ⁻⁹ m | Predicts shear and normal separation | | |
| 1990 Tvergaard | | δ,,δ, = critical displacements. σ,, = Max. Stress | Interfaces of whisker-reinforced metal matrix composites | $\delta_{n} = \delta_{+} = 1 \times 10^{9} \text{m.}$ E=60GPa(Young's mod) $\sigma_{y}/E = 0.005$ $\sigma_{max}/\sigma_{y} = 5 - 9$ | Linear Model | | |
| 1996 Camacho and Ortiz | σ_{0} σ_{1} σ_{2} σ_{1} σ_{1} σ_{2} σ_{1} σ_{2} σ_{1} σ_{1} σ_{2} σ_{1} σ_{1} σ_{2} σ_{1} σ_{1} σ_{2} σ_{1} σ_{2} σ_{1} σ_{1} σ_{2} σ_{1} σ_{1} σ_{1} σ_{1} σ_{2} σ_{1} σ_{1 | σ_0 , $\tau_0 = Nor.$ and Shr stress at fracture initiation $\delta_{\sigma^{cr}}$, $\delta_{\tau^{cr}} = crit.$ nor.opening and shr. opening displ. $G_c = Fracture energy$ | Im pact | A lumina: $\mathcal{O}_0 = 400 \text{ MPa}$ $\mathcal{S}_{acr} = 1.7 \times 10^{-7} m$ Steel: $\sigma_0 = 1500 MPa$ $\mathcal{S}_{acr} = 2.7 \times 10^{-4} m$ | Predicts failure by both shear and normal separation in tension and by shear separation in compression | | |
| 1997 Gaubelle & Bayler | | Φ_n, Φ_t : Work of normal and tangential separation Δ_n, Δ_t : Normal and Tangential displacement jump δ_n, δ_t : Normal and tangential interface characteristic- length | de lam ination by low-ve locity impact | σ _{max} = E/10 Critical normal displacement jum p Δ _{nc} =10 [*] ~10 [*] m | Bilinear model Ascending curve can be matched to initial stiffness of the material | | |
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Scales of cohesive model

 $\tilde{\phi} = \tilde{\sigma}\tilde{\delta}$ Energy 2 out of the three are independent $\tilde{\varphi} = \sigma \sigma$ $\tilde{p} = \rho \tilde{v} = \frac{\tilde{\sigma}}{c_{\rm d}}$ $\tilde{E} = \frac{\tilde{v}}{c_{\rm d}} = \frac{\tilde{\sigma}}{\rho c_{\rm d}^2} \propto \frac{\tilde{\sigma}}{\|\mathbf{C}\|}$ Linear momentum Strain $\tilde{v} = \frac{\tilde{\delta}}{\tilde{\tau}} = \frac{\tilde{\sigma}}{\rho c_{\rm d}}$ Velocity $\tilde{L} = c_{\rm d}\tilde{\tau} = \frac{\rho c_{\rm d}^2 \tilde{\delta}}{\tilde{\sigma}} \propto \Lambda^0$ $\tilde{\tau} = \frac{\rho c_{\rm d} \tilde{\delta}}{\tilde{\sigma}}$ Process zone size Λ Length Time Influences time step for time marching methods

Why process zone size is important?

- Importance of process zone size Λ
 - Static estimate:

$$\begin{split} \Lambda &= \varsigma \pi \frac{\mu}{1-\nu} \frac{\dot{\phi}}{\tilde{\sigma}^2} \propto \tilde{L} \\ \varsigma &= \begin{cases} \frac{1}{4} & \text{Dugdale model} \\ \frac{9}{16} & \text{Potential-based TSRs} \end{cases} \end{split}$$



- Minimum number of elements in process zone size: There should be at least 4-10 elements along the PZ
- Dynamic estimate: PZS decreases as crack speed \hat{v} approaches Rayleigh wave speed $c_{\rm R}$

$$A(\hat{v}) = \frac{A}{A(\hat{v})}, \quad A(\hat{v}) \to 0 \text{ as } \hat{v} \to c_{\mathrm{R}}$$

Smaller elements are needed in PZT as crack accelerates!



When LEFM results match cohesive solutions?



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steady state

0.2

0.4

à'

0.6

0.8

1

'n

0L 0

2

4

t'

6

8

10

Resistance (fracture toughness) versus work of separation

- Fracture toughness (Γ): LEFM: Energy needed to create one unit surface of crack
- Work of separation (φ): CFM: Energy needed to entirely debond a point in time per area (following a traction-separation-relation)
 - Relation between ϕ and G:

$$G = \frac{1}{\hat{v}} \int_{-\Lambda_{(k)}}^{0} \tilde{s}(\delta_k) \frac{\partial \delta_k}{\partial t} \, \mathrm{d}x + \int_{0}^{\delta_{\mathrm{T}}} \tilde{s}(\delta_k) \, \mathrm{d}\delta_k = I_t + \tilde{\phi}_{(k)}$$

 $\left|\frac{I_t}{\tilde{\phi}_{(k)}}\right| \leq \frac{\tilde{\sigma}_{(k)}A_{(k)}}{\hat{v}\tilde{\phi}_{(k)}} \left|\frac{\partial \delta_k}{\partial t}\right|_{\infty} \approx \frac{\zeta_{(k)}\pi\mu}{(1-\nu)\hat{v}A_{(k)}(\hat{v})\tilde{\sigma}_{(k)}} \left|\frac{\partial \delta_k}{\partial t}\right|_{\infty}.$

- Dynamic part () goes to zero when:
 - Steady state crack propagation (crack speed does not change).
 - When the crack speed tends to Rayleigh wave speed (c_R)



Artificial compliance for intrinsic cohesive models

- Artificial compliance becomes important if <u>cohesive surfaces are added between</u> all elements for intrinsic models to find crack propagation path.
- The artifical compliance is computed as,

 $\begin{array}{rcl} \Delta &=& \Delta_e + \Delta_c, & \Delta_e = \mbox{ elastic displacement}, \Delta_c = \mbox{ cohesive separation} \\ \hline \frac{\sigma}{E_{\rm eff}} h &=& \frac{\sigma}{E} h + \frac{\sigma}{K} & \Rightarrow \\ \hline \frac{1}{E_{\rm eff}} &=& \frac{1}{E} + \frac{1}{Kh} & \Rightarrow \end{array}$

Artificial compliance is,

$$\begin{split} C_c &= \frac{1}{Kh} = \frac{\tilde{\delta}}{\tilde{\sigma}h} = \frac{1}{E_c}, \text{where} \\ E_c &= Kh = \frac{\tilde{\sigma}h}{\tilde{\delta}}, \text{and effective elastic modulus is} \\ \frac{1}{E_{\text{eff}}} &= \frac{1}{E} + \frac{1}{E_c}, \quad \Rightarrow E_{\text{eff}} = \frac{EE_c}{E + E_c} \end{split}$$

- That is the smaller element spacing h or softer the initial slope K of TSR the higher artificial compliance (higher errors)
- While <u>extrinsic</u> cohesive models do not have the same problem, adaptive insertion of cohesive surfaces is more challenging for them.





4.3 Mixed mode fracture

- 4.3.1 Crack propagation criteria
 - a) Maximum Circumferential Tensile Stress
 - b) Maximum Energy Release Rate
 - c) Minimum Strain Energy Density
- 4.3.2 Crack Nucleation criteria





$$F_{yld}(\sigma_1, \sigma_2, \sigma_y) = 0 \qquad \sigma_v = \sigma_y \text{ for } \sigma_v = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}}$$

Example: von Mises yield criterion

Motivation: Experiment verification of the mixed-mode failure criterion



Data points do not fall exactly on the circle.

$$\left(\frac{K_I}{K_{Ic}}\right)^2 + \left(\frac{K_{II}}{K_{IIc}}\right)^2 = 1 \quad \text{self-similar growth} \quad \boxed{G = \frac{(\kappa + 1)^2}{8\mu}}$$

Mixed-mode crack growth

Combination of mode-I, mode-II and mode-III loadings: mixed-mode loading.

Cracks will generally propagate along a curved surface as the crack seeks out its path of **least resistance**.

Only a 2D mixed-mode loading (mode-I and mode-II) is discussed.

4.3 Mixed mode fracture

- 4.3.1 Crack propagation criteria
 - a) Maximum Circumferential Tensile Stress
 - b) Maximum Energy Release Rate
 - c) Minimum Strain Energy Density
- 4.3.2 Crack Nucleation criteria

Erdogan and Sih



the direction of propagation is given by the angle θ_c for which

(from M. Jirasek)

maximum circumferential stress criterion (maximum hoop stress criterion): crack propagates in the direction perpendicular to the maximum circumferential stress (evaluated on a circle of a small diameter centered at the tip)

$$\sigma_{\theta}(r,\theta_{c}) = \max_{-\pi < \theta < \pi} \sigma_{\theta}(r,\theta)$$
principal stress
$$\tau_{1}$$

$$\sigma_r = \frac{K_{\rm I}}{\sqrt{2\pi r}} \left(\frac{5}{4} \cos\frac{\theta}{2} - \frac{1}{4} \cos\frac{3\theta}{2}\right) + \frac{K_{\rm II}}{\sqrt{2\pi r}} \left(-\frac{5}{4} \sin\frac{\theta}{2} + \frac{3}{4} \sin\frac{3\theta}{2}\right)$$
(7.35a)

$$\sigma_{\theta} = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \left(\frac{3}{4} \cos \frac{\theta}{2} + \frac{1}{4} \cos \frac{3\theta}{2}\right) + \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}} \left(-\frac{3}{4} \sin \frac{\theta}{2} - \frac{3}{4} \sin \frac{3\theta}{2}\right)$$
(7.35b)

$$\tau_{r\theta} = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \left(\frac{1}{4} \sin \frac{\theta}{2} + \frac{1}{4} \sin \frac{3\theta}{2} \right) + \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}} \left(\frac{1}{4} \cos \frac{\theta}{2} + \frac{3}{4} \cos \frac{3\theta}{2} \right) .$$

$$(7.35c)$$

$$\tau_{r\theta} = 0 \longrightarrow K_{I} \left(\sin \frac{\theta}{2} + \sin \frac{3\theta}{2} \right) + K_{II} \left(\cos \frac{\theta}{2} + 3 \cos \frac{3\theta}{2} \right) = 0$$

$$\theta_c = 2 \arctan \frac{1}{4} \left(K_I / K_{II} \pm \sqrt{(K_I / K_I I)^2 + 8} \right)$$



$$\tan\frac{\theta_{0}}{2} = \frac{1}{4}\frac{K_{\rm I}}{K_{\rm II}} \pm \frac{1}{4}\sqrt{\left(\frac{K_{\rm I}}{K_{\rm II}}\right)^{2} + 8}$$

Maximum allowable traction $\sigma_{\theta max}$ is reached at angle $\theta = \theta max$ and distance from crack tip r_0 : $\sigma_{\theta max} \sqrt{2\pi r_0} = K_{\rm Ic} = \cos \frac{\theta_0}{2} \left[K_{\rm I} \cos^2 \frac{\theta_0}{2} - \frac{3}{2} K_{\rm II} \sin \theta_0 \right]$

Fracture criterion $K_{eq} \ge K_{Ic}$

$$\sigma_{\theta} = \frac{K_{\rm I}}{\sqrt{2\pi r}} \cos\frac{\theta_0}{2} \left(1 - \sin^2\frac{\theta_0}{2}\right) + \frac{K_{\rm II}}{\sqrt{2\pi r}} \left(-\frac{3}{4}\sin\frac{\theta_0}{2} - \frac{3}{4}\sin\frac{3\theta_0}{2}\right) \tag{7.9}$$

must reach a critical value which is obtained by rearranging the previous equation

$$\sigma_{\theta max}\sqrt{2\pi r} = K_{\rm Ic} = \cos\frac{\theta_0}{2} \left[K_{\rm I}\cos^2\frac{\theta_0}{2} - \frac{3}{2}K_{\rm II}\sin\theta_0 \right]$$
(7.10)

which can be normalized as

$$\frac{K_{\rm I}}{K_{\rm Ic}} \cos^3 \frac{\theta_0}{2} - \frac{3}{2} \frac{K_{\rm II}}{K_{\rm Ic}} \cos \frac{\theta_0}{2} \sin \theta_0 = 1$$
(7.11)

¹¹ This equation can be used to define an equivalent stress intensity factor K_{eq} for mixed mode problems

$$K_{eq} = K_{\rm I} \cos^3 \frac{\theta_0}{2} - \frac{3}{2} K_{\rm II} \cos \frac{\theta_0}{2} \sin \theta_0$$
(7.12)





XFEM



$$\theta_c = 2 \arctan \frac{1}{4} \left(K_I / K_{II} \pm \sqrt{(K_I / K_I I)^2 + 8} \right)$$

Modifications to maximum circumferential stress criterion

• Effective traction can be defined as a function of both normal σ_{θ} and tangential $\tau r\theta$ components of traction. For example:

$$\sigma_{\text{eff}} = \sqrt{\sigma_{\theta}^2 + (\alpha \tau_{r\theta})^2}$$

combines normal and tangential components through mode mixity parameter α .

• Crack propagation direction θ_c can be based on maximizing effective traction:

$$\sigma_{\rm eff}(r,\theta_c) = \max_{-\pi < \theta < \pi} \sigma_{\rm eff}(r,\theta)$$

 For example, in soil and rock applications normal tractions can be compressive for cracks that propagate under high shear tractions.



4.3 Mixed mode fracture

4.3.1 Crack propagation criteria

a) Maximum Circumferential Tensile Stress

b) Maximum Energy Release Rate

c) Minimum Strain Energy Density

4.3.2 Crack Nucleation criteria

Maximum Energy Release Rate



G: crack driving force -> crack will grow in the direction that G is maximum

Erdogan, F. and Sih, G.C. 1963

"If we accept Griffith (energy) theory as the valid criteria which explains crack growth, then the crack will grow in the direction along which the elastic energy release per unit crack extension will be maximum and the crack will start to grow when this energy reaches a critical value (or $G = G(\delta, \theta)$). Evaluation of $G(\delta, \theta)$ poses insurmountable mathematical difficulties."

Maximum Energy Release Rate



 $\left[(1+3\cos^2\theta)K_I^2+8\sin\theta\cos\theta K_IK_{II}+(9-5\cos^2\theta)K_{II}^2\right]$

Maximum Energy Release Rate

Maximization condition

$$\frac{\partial G(\theta)}{\partial \theta} = 0$$

$$\frac{\partial^2 G(\theta)}{\partial \theta^2} < 0$$

$$K_{\rm I}, K_{\rm II}$$

$$K_{\rm I}, K_{\rm II}$$

$$G(\theta) = \frac{4}{E'} \left(\frac{1}{3 + \cos^2 \theta}\right)^2 \left(\frac{1 - \frac{\theta}{\pi}}{1 + \frac{\theta}{\pi}}\right)^{\frac{\theta}{\pi}}$$
$$[(1 + 3\cos^2 \theta)K_I^2 + 8\sin\theta\cos\theta K_I K_{II} + (9 - 5\cos^2 \theta)K_{II}^2]$$

$$4\left(\frac{1}{3+\cos^2\theta_0}\right)^2 \left(\frac{1-\frac{\theta_0}{\pi}}{1+\frac{\theta_0}{\pi}}\right)^{\frac{\theta_0}{\pi}} \left[\left(1+3\cos^2\theta_0\right)\left(\frac{K_{\rm I}}{K_{\rm Ic}}\right)^2+8\sin\theta_0\cos\theta_0\left(\frac{K_{\rm I}K_{\rm II}}{K_{\rm Ic}^2}\right)+\left(9-5\cos^2\theta_0\right)\left(\frac{K_{\rm II}}{K_{\rm Ic}}\right)^2\right]=1$$

4.3 Mixed mode fracture

4.3.1 Crack propagation criteria

a) Maximum Circumferential Tensile Stress

- b) Maximum Energy Release Rate
- c) Minimum Strain Energy Density
- 4.3.2 Crack Nucleation criteria

Strain Energy Density (SED) criterion Sih 1973

$$U_i = \int_0^{\epsilon_{ij}} \sigma_{ij} d\epsilon_{ij} \quad U_i = \frac{1}{4\mu} \left[\frac{\kappa + 1}{4} (\sigma_x^2 + \sigma_y^2) - 2(\sigma_x \sigma_y - \tau_{xy}^2) \right]$$

$$\begin{split} \sigma_x &= \frac{K_{\rm I}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) - \frac{K_{\rm II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \left(2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right) \\ \sigma_y &= \frac{K_{\rm I}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) + \frac{K_{\rm II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \quad (7.13) \\ \tau_{xy} &= \frac{K_{\rm I}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2} + \frac{K_{\rm II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \,. \end{split}$$

Strain Energy Density (SED) criterion

- Crack direction θ_0 which minimizes the strain energy density S
- Crack Extends when S reaches a critical value at a distance r_0

Minimization condition

Pure mode I (0 degree has smallest S)



$$\frac{\partial S}{\partial \theta} = 0$$
$$\frac{\partial^2 S}{\partial \theta^2} > 0$$

Strain Energy Density (SED) criterion

$$\frac{8\mu}{(\kappa-1)} \left[a_{11} \left(\frac{K_{\mathrm{I}}}{K_{\mathrm{Ic}}} \right)^2 + 2a_{12} \left(\frac{K_{\mathrm{I}}K_{\mathrm{II}}}{K_{\mathrm{Ic}}^2} \right) + a_{22} \left(\frac{K_{\mathrm{II}}}{K_{\mathrm{Ic}}} \right)^2 \right] = 1$$

$$a_{11} = \frac{1}{16\mu} \left[(1 + \cos\theta) \left(\kappa - \cos\theta\right) \right]$$

$$a_{12} = \frac{\sin\theta}{16\mu} \left[2\cos\theta - (\kappa - 1) \right]$$

$$a_{22} = \frac{1}{16\mu} \left[(\kappa + 1) \left(1 - \cos\theta \right) + (1 + \cos\theta) \left(3\cos\theta - 1 \right) \right]$$

$$\kappa = \frac{3-\nu}{1+\nu} \quad \text{(plane stress)}$$

 $\kappa = 3 - 4\nu \quad \text{(plane strain)}$

Comparison: a) Crack Extension angle

Angle of Crack Propagation Under Mixed Mode Loading



Good agreement for low K_{II}

Comparison: b) Locus of crack propagation



Mixed mode criteria: Observations

- 1. First crack extension θ_0 is obtained followed by on whether crack extends in θ_0 direction or not.
- 2. Strain Energy Density (SED) and Maximum Circumferential Tensile Stress require an r₀ but the final crack propagation locus is independent of r₀.
- 3. SED theory depends on Poisson ratio v.
- 4. All three theories give identical results for small ratios of K_{II}/K_I and diverge slightly as this ratio increases
- 5. Crack will always extend in the direction which attempts to minimize K_{II}/K_{I} .
- 6. For practical purposes during crack propagation all three theories yield very similar paths as from 4 and 5 cracks extend mostly in mode I where the there is a better agreement between different criteria

Implication on close to mode I crack propagation

Criterion of local symmetry

When cracks grow in non-uniform stress fields, the path of the fracture is generally curved. The path taken by a crack in brittle homogeneous isotropic material can be assumed to be one for which the local stress field at the tip is of a Mode I type*:

$K_{II} = 0$ at the tip

*Cotterell, Brian, and JRf Rice. "Slightly curved or kinked cracks." International journal of fracture 16.2 (1980): 155-169

[1] N.V. Banichuk, "Determination of the Form of a Curvilinear Crack by Small Parameter Technique" Izv. An SSR, MTT 7, 2 (1970) 130-7 (in Russian).

[2] R.V. Goldstein and R.L. Salganik, "Plane Problem of Curvilinear Cracks in an Elastic Solid", Izv. An SSR, MIT 7, 3 (1970) 69-82 (in Russian).

[3] R.V. Goldstein and R.L. Salganik, International Journal of Fracture 10 (1974) 507-23.

If $K_{II} \neq 0$ at the crack tip, the crack would abruptly change its direction; the crack path would exhibit a kink at this position, and hence it would not be smooth [21]. By driving a curved crack straightly in the tangent direction at the crack tip the local symmetry (2) will be broken ($K_{II} \neq 0$). The change of $K_{II}(s)$ along a straight extension *s* in the tangent direction determines the curvature *C* of the crack path according to Amestoy and Leblond [see Eq. (104) in Ref. [25]]

$$C = -\frac{2}{K_{\rm I}} \left[\frac{dK_{\rm II}(s)}{ds} \right]_{\rm straight}.$$
 (3)

Van-Bac Pham, Hans-Achim Bahr, Ute Bahr, Herbert Balke, Hans-Jürgen Weiss, Global bifurcation criterion for oscillatory crack path instability, PHYSICAL REVIEW E 77, 066114 (2008)

- [21] B. Cotterell and J. R. Rice, Int. J. Fract. 16, 155 (1980)
- [25] M. Amestoy and J. B. Leblond, Int. J. Solids Struct. 29, 465 (1992)

Example of critical local symmetry



6.2 (1980): 155-169

FIG. 1. Experiment by Yuse and Sano [1].

Van-Bac Pham, Hans-Achim Bahr, Ute Bahr, Herbert Balke, Hans-Jürgen Weiss, Global bifurcation criterion for oscillatory crack path instability, PHYSICAL REVIEW E 77, 066114 (2008)

4.3 Mixed mode fracture

- 4.3.1 Crack propagation criteria
 - a) Maximum Circumferential Tensile Stress
 - b) Maximum Energy Release Rate
 - c) Minimum Strain Energy Density
- 4.3.2 Crack Nucleation criteria

Crack nucleation criterion

- Cracks nucleate from microscopic material defects under high stress/ strain loads.
- For each crack propagation criterion there can be a corresponding nucleation criterion.
- For example for maximum circumferential tensile stress, a crack nucleates when the maximum principle stress σ₁ at a point reaches material strength σ₀:

$$\max_{-\pi < \theta < \pi} \sigma_{\theta}(r \to 0^+, \theta) = \sigma_1 = \sigma_0$$
, crack nucleates

Although we assume that there is no initial crack tip, we can measure r relative to the potential nucleation point.

• Same concept applies to modified maximum circumferential tensile stress criteria:

$$\max_{-\pi < \theta < \pi} \sigma_{\text{eff}}(r \to 0^+, \theta) = \sigma_0$$
, crack nucleates



Crack nucleation criterion

 For Maximum Energy Release Rate Criterion if we assume there are no defects, there will be no crack nucleation. However, assuming that local stress field generates a tensile maximum principal stress of σ₁ a "microscopic" initial crack (defect) of length a_{ini} perpendicular to σ₁ direction generates,

$$G = \frac{K_I^2 + K_{II}^2}{E'} = \pi a_{\rm ini} \sigma_1^2$$

so the microcrack propagates (*i.e.*, a "macroscopic" crack nucleates) when,

$$G = G_c \quad \Leftrightarrow \quad \sigma_1 = \sqrt{\frac{G_c}{\pi a_{\text{ini}}}}$$

- Initial crack direction perpendicular to σ_1 is chosen to maximize G.
- We have assumed the initial crack to be small enough to use the infinite domain SIF formula of $K_I = \sqrt{\pi a \bar{\sigma}}$.

8. Fatigue

- 8.1. Fatigue regimes
- 8.2. S-N, P-S-N curves
- 8.3. Fatigue crack growth models (Paris law)

- Fatigue life prediction

- 8.4. Variable and random load
 - Crack retardation due to overload
Fatigue examples

Key Idea: Fluctuating loads are more dangerous than monotonic loads.

Example: Comet Airliner (case study). The actual cabin pressure differential when the plane was in flight was ≈ 8.5 pounds per square inch (psi). The design pressure was ≈ 20 psi (a factor of safety greater than 2). Thought to be safe! However, crack growth due to *cyclic* loading caused catastrophic failure of the aircraft.

Fatigue fracture is prevalent!

- Deliberately applied load reversals (e.g. rotating systems)
- Vibrations (machine parts)
- Repeated pressurization and depressurization (airplanes)
- Thermal cycling (switching off electronic devices)
- Random forces (ships, vehicles, planes) (source: Schreurs fracture notes 2012)

Fatigue occurs always and everywhere and is a major source of mechanical failure

Fatigue

- Fatigue occurs when a material is subjected to repeated loading and unloading (cyclic loading).
- Under cyclic loadings, materials can fail (due to fatigue) at stress levels well below their yield strength or crack propagation limit-> fatigue failure.
- <u>ASTM</u> defines *fatigue life*, *N_f*, as the number of stress cycles of a specified character that a specimen sustains before <u>failure</u> of a specified nature occurs.



Fatigue striations



Fracture surface of a 2024-T3 aluminum alloy (source S. Suresh MIT)

Striation caused by individual microscale crack advance incidents

Fatigue crack growth:

Microcrack formation in accumulated slip bands due to repeated loading





Fatigue Regimes

| Fatigue | Failure cycles N _R | Pertinent stress | Strain ratio $\Delta \varepsilon^p / \Delta \varepsilon^e$ | Energy ratio $\Delta W^p / \Delta W^e$ |
|---|--|---|--|--|
| Very high cycle fatigue | > 10 ⁷ | $<\sigma_F$ | ≈ 0 | pprox 0 |
| High cycle fatigue | 10^5 to 10^6 | <σγ | ≈ 0 | pprox 0 |
| Low cycle fatigue | 10^2 to 10^4 | $\sigma \gamma$ to σU | 1 to 10 | 1 to 10 |
| Very low cycle fatigue | 1 to 20 | $\approx \sigma U$ | 10 to 100 | 10 to 100 |
| Very high cycle and hi Stresses are well b | <mark>gh cycle fatigue</mark> : below yield/ultin | nate strength | | |
| Very high cycle and hi Stresses are well b There is almost no Fatigue models ba Stress-life approad Low cycle and very low | gh cycle fatigue: below yield/ultin plastic deforma sed on LEFM the ches are used (st w cycle fatigue: | nate strength ation (in term eory (e.g. SIF cress-centered | s of strain and en K) are applicable. d criteria) | ergy ratios) |
| Very high cycle and hi Stresses are well b There is almost no Fatigue models ba Stress-life approad Low cycle and very low Stresses are in the | gh cycle fatigue: below yield/ultin o plastic deforma sed on LEFM the ches are used (st w cycle fatigue: order of yield/u | nate strength ation (in term eory (e.g. SIF cress-centered | s of strain and en K) are applicable. d criteria) | ergy ratios) |
| Very high cycle and hi Stresses are well b There is almost no Fatigue models ba Stress-life approad Low cycle and very low Stresses are in the There is consideral | gh cycle fatigue: below yield/ultin o plastic deformation sed on LEFM the ches are used (st w cycle fatigue: order of yield/u ble plastic defor | nate strength ation (in term eory (e.g. SIF cress-centered ultimate stren | s of strain and en K) are applicable. d criteria) ngth. | ergy ratios) |

• Strain-life approaches are used (strain-centered criteria)

Cyclic loadings



Cyclic vs. static loadings

- Static: Until K reaches K_c, crack will not grow
- Cyclic: K applied can be well below K_c, crack still grows!!!
- 1961, Paris Erdogan used the theory of LEFM to explain fatigue cracking successfully.
- Methodology: experiments first, then empirical equations are proposed.



- 1. Initially, crack growth rate is small
- 2. Crack growth rate increases rapidly when a is large
- 3. Crack growth rate increases as the applied stress increases

S-N curve

Reminder: <u>ASTM</u> defines *fatigue life*, N_f , as the number of stress cycles of a specified character that a specimen sustains before <u>failure</u> of a specified nature occurs.



endurance limit (g.han keo dai)

S-N-P curve: scatter effects



Effect of mean stress



SUS

r = 1 is called Goodman line which is close to the results of notched specimens.

r = 2 is the Gerber parabola which better represents ductile metals.

Soderberg (1939)

 $\frac{\sigma_a^*}{\sigma_a} = 1 - \frac{\sigma_m}{\sigma_{y0}}$

Constant variable cyclic load ssy



$$R = K_{
m min}/K_{
m max}$$
 crack growth rate $\Delta K = K_{
m max} - K_{
m min}$

$$\Delta K = K_{\max} - K_{\min} = K_{\max} \left(1 - R \right)$$

$$\overbrace{\frac{da}{dN} \neq f_1 \land \Delta K, R} = f_1 \land \Delta K, R$$
crack growth models





Fatigue crack growth stages



Fatigue crack growth stages

| Regime | A | В | С |
|----------------------------------|-----------------------------------|---------------------------------------|---|
| Terminology | Slow-growth rate (near-threshold) | Mid-growth rate (Paris regime) | High-growth rate |
| Microscopic failure mode | Stage I, single shear | Stage II, (striations) duplex slip | Additional static modes |
| Fracture surface features | Faceted or serrated | Planar with ripples | Additional cleavage or microvoid coalescence |
| Crack closure levels | High | Low | |
| Microstructural effects | Large | Small | Large |
| Load ratio effects | Large | Small | Large |
| Environmental effects | Large | | Small |
| Stress state effects | | Large | Large |
| Near-tip plasticity [†] | $r_c \leq d_g$ | $r_a \geq d_g$ | $r_c \gg d_g$ |

large influence on crack growth for certain combinations of environment, load ratio and frequency.

re and dg refer to the cyclic plastic zone size and the grain size, respectively.

Paris' law

not depends on load ratio R

$$\frac{da}{dN} = C(\Delta K)^m, \ \Delta K = K_{\max} - K_{\min}$$

Table 1: Numerical parameters in the Paris equation.

| alloy | m | A |
|----------|-----|-------------------|
| Steel | 3 | 10^{-11} |
| Aluminum | 3 | 10^{-12} |
| Nickel | 3.3 | $4	imes 10^{-12}$ |
| Titanium | 5 | 10 ⁻¹¹ |

C,m

are material properties that must be determined experimentally from a log(delta K)-log(da/dN) plot. m 2-4 metals 4-100 ceramics/ polymers

Other fatigue models



Crack closure

- At low loads, the compliance of cracked specimens are close to that of un-cracked specimens.

- Contact of crack faces: crack closure

- Fatigue crack growth occurs only when crack is fully open.



$$\Delta K_{\text{eff}} = K_{\text{max}} - K_{\text{op}}$$

K_{op}: opening SIF



Tension/compression cyclic loads

$$R = \frac{\sigma_{\min}}{\sigma_{\max}} < 0$$

$$\frac{da}{dN} = C(\Delta K)^m, \ \Delta K = K_{\max} - K_{\min}$$



Fully Reversed Loading

Fatigue life calculation

When we are in regime B (*Paris* regime) the following calculation can be carried out to determine the number of cycles to failure. From the Paris Law:

$$\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}\boldsymbol{N}} = \boldsymbol{C}\left(\Delta \boldsymbol{K}\right)^{\boldsymbol{m}}$$

 ΔK can be expressed in terms of $\Delta \sigma$

 $\Delta K = Y \Delta \sigma \sqrt{\pi a}$

Where Y depends the specific specimen geometry.

Fatigue life calculation

Thus the Paris Law becomes:

$$rac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}\boldsymbol{N}} = \boldsymbol{C}\left(\boldsymbol{Y}\Delta\boldsymbol{\sigma}\sqrt{\pi\boldsymbol{a}}
ight)^{\boldsymbol{m}}$$

Assume that Y is a constant. Solve for da and integrate both sides:

$$\int_{a_0}^{a_f} \frac{\mathrm{d}a}{a^{m/2}} = CY^m (\Delta\sigma)^m \pi^{m/2} \int_0^{N_f} \mathrm{d}N$$

Fatigue life calculation



Fatigue life calculation: Initial crack length a₀

In these expressions, we need to determine the initial crack length a_0 and the final crack length a_f (sometimes called the critical crack length).

How do we determine the initial crack length a₀?

Cracks can be detected using a variety of techniques, ranging from simple visual inspection to more sophisticated techniques based on ultrasonics or x-rays. If no cracks are detectable by our inspection, we must assume that a crack just at the resolution of our detection system exists.

Nondestructive testing (NDT)

Nondestructive Evaluation (NDE), nondestructive Inspection (NDI)

NDT is a wide group of analysis techniques used in science and industry to evaluate the properties of a material, component or system without causing damage

NDT: provides input (e.g. crack size) to fracture analysis



Fatigue life calculation: Final crack length a_f

How do we determine the final crack length a_f ? We know that eventually the crack can grow to a length at which the material fails immediately, i.e.



or

 $Y\sigma_{\max}\sqrt{\pi a_f} \to K_c$

Fatigue life calculation: Crack-tolerant design

Thus we may solve for a_f as follows:

 $a_f = \frac{1}{\pi} \frac{K_c^2}{Y^2 \sigma_{\max}^2}$

A very important idea that comes from this analysis is the following: even if a component has a detectable crack, it need not be removed from service! Using this framework, the remaining life can be assessed. The component can remain in service provided it is inspected periodically. This is the **crack-tolerant** or **damage tolerant** design approach.

| Fatigue life calculation: |
|--|
| Example |
| • Given: Griffith crack, $2a_0, \Delta\sigma, C, m, K_{Ic}, N_0$ |
| • Question: compute N_f $K = \sigma \sqrt{\pi a}$ |
| $\begin{array}{ll} \text{analytical} \\ \text{integration} \\ \hline m=4 \end{array} \qquad $ |
| $N = N_0 + \frac{1}{C(\Delta\sigma)^4 \pi^2} \int_{a_0}^{a_f} \frac{da}{a^2} = N_0 + \frac{1}{C(\Delta\sigma)^4 \pi^2} \left(\frac{1}{a_0} - \frac{1}{a_f}\right)$ |
| measurement $K_{\max} = \sigma_{\max} \sqrt{\pi a_f} = K_{Ic}$ |

Numerical integration of fatigue law

$$N = N_0 + \int_{a_0}^{a_f} rac{da}{C(\Delta\sigma f(a/W)\sqrt{\pi a})^m}$$
 tedious to compute

set
$$\Delta \sigma$$
, ΔN , a_c
initialize $N = 0$, $a = a_0$
while $a < a_c$
 $\Delta K = \beta \Delta \sigma \sqrt{\pi * a}$
 $\frac{da}{dN} = C * (\Delta K)^m \rightarrow \Delta a = \frac{da}{dN} * \Delta N$
 $a = a + \Delta a$
 $N = N + \Delta N$
end

Importance of initial crack length



Fatigue design philosophies: Safe-life approach

- Determine typical service spectra.
- Estimate useful fatigue life based on laboratory tests or analyses.
- Add factor of safety.
- At the end of the expected life, the component is retired from service, even if no failure has occurred and the component has considerable residual life.
- Emphasis on prevention of crack initiation.
- Approach is theoretical in nature.

Fatigue design philosophies: Fail-safe approach

- Even if an individual member of a component fails, there should be sufficient structural integrity to operate safely.
- Multiple load paths and crack arresters.
- Mandates periodic inspection.
- Accent on crack growth rather than crack initiation.

Fatigue design philosophies: Case study

F-100 gas turbine engines in F-15 and F-16 fighter aircraft for U.S. Airforce.

- Old Approach:
- 1000 disks could be retired from service when, statistically, only one disk had a fatigue crack (a < 0.75 mm).

New, RFC Approach (since 1986):

- Retirement of a component occurs when the unique fatigue life of that particular component is expended.
- Retirement only when there is reason for removal (e.g. crack).

Fatigue design philosophies: Case study

- Twin engine F-15 and single engine F-16.
- 3200 engines in the operational inventory of U.S. Air Force.
- 23 components of the engine are managed under RFC.
- 1986-2005 life cycle cost savings: \$1,000,000,000
- Additional labor and fuel costs: \$655,000,000

Miner's rule for variable load amplitudes 1945



Influence of sequence of loading

The component is assumed to fail when the total damage becomes equal to 1, or



It is assumed that the **sequence** in which the loads are applied has no influence on the lifetime of the component. In fact, the sequence of loads *can* have a large influence on the lifetime of the component.
Influence of sequence of loading

Consider a sequence of two different cyclic loads, σ_{a1} and σ_{a2} . Let $\sigma_{a1} > \sigma_{a2}$.

Case 1: Apply *o*_{a1} then *o*_{a2}.

In this case, $\sum_{i} \frac{n_{i}}{N_{fi}}$ can be less than 1. During the first loading (σ_{a1}) numerous microcracks can be initiated, which can be further propagated by the second loading (σ_{a2}) .

Influence of sequence of loading

Case 2: Apply σ_{a2} then σ_{a1} .

In this case $\sum_{i} \frac{n_{i}}{N_{fi}}$ can be greater than 1. The first loading (σ_{a2}) is not high enough to cause any microcracks, but it is high enough to strain harden the material. Then in the second loading (σ_{a1}), since the material has been hardened it is more difficult to initiate any damage in the material.

Variable amplitude cyclic loadings



Overload and crack retardation

It was recognized empirically that the application of a tensile overload in a constant amplitude cyclic load leads to crack retardation following the overload; that is, the crack growth rate is smaller than it would have been under constant amplitude loading.



Crack retardation



Point A: plastic point B: elastic



After unloading: point A and B has more or less the same strain -> point A : compressive stress.

Crack retardation



a large plastic zone at overload has left behind

residual compressive plastic zone

close the crack->crack retards

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Overload and crack retardation

It was recognized empirically that the application of a tensile overload in a constant amplitude cyclic load leads to crack retardation following the overload; that is, the crack growth rate is smaller than it would have been under constant amplitude loading.



Shot peening is a cold working process in which the surface of a part is bombarded with small spherical media called *shot*. Each piece of shot striking the surface acts as a tiny peening hammer, imparting to the surface a small indentation or dimple. The net result is a layer of material in a state of residual compression. It is well established that cracks will not initiate or propagate in a compressively stressed zone.

A typical residual stress profile created by shot peening is shown below:



Since nearly all fatigue and stress corrosion failures originate at the surface of a part, compressive stresses induced by shot peening provide *considerable* increases in part life. Typically the residual stress produced is at least half the yield strength of the material being peened.

The benefits of shot peening are a result of the *residual* compressive stress and the cold working of the surface.

Residual stress: Increases resistance to fatigue crack growth, corrosion fatigue, stress corrosion cracking, hydrogen assisted cracking, fretting, galling and erosion caused by cavitation.

Cold Working: Benefits include work hardening (strengthening), intergranular corrosion resistance, surface texturing, closing of porosity and testing the bond of coatings.

Residual stresses are those stresses remaining in a part after all manufacturing operations are completed, and with no external load applied. In most applications for shot peening, the benefit obtained is the direct result of the residual compressive stress produced.

Damage tolerance design 1970s

(stress concentration: possible crack sites)

- 1. Determine the size of initial defects $, a_0$ NDI
- 2. Calculate the critical crack size at_which failure would occur $\sigma \sqrt{\pi a_c} = K_{Ic}$
- Integrate the fatigue crack growth equations to compute the number of load cycles for the crack to grow from initial size to the critical size

$$N = N_0 + \int_{a_0}^{a_c} \frac{da}{C(\Delta\sigma\sqrt{\pi a})^m}$$

4. Set inspection intervals

Examples for Fatigue

A large plate contains a crack of length $2a_0$ and is subjected to a constant-amplitude tensile cyclic stress normal to the crack which varies between 100 MPa and 200 MPa. The following data were obtained: for $2a_0 = 2$ mm it was found that N = 20,000cycles were required to grow the crack to $2a_f = 2.2$ mm, while for 2a = 20 mm it was found that N = 1000 cycles were required to grow the crack to $2a_f = 22$ mm. The critical stress intensity factor is $K_c = 60$ MPa \sqrt{m} . Determine the constants in the Paris (Equation (9.3)) and Formam (Equation (9.4)) equations.



Forman's model

$$R = \frac{\sigma_{\min}}{\sigma_{\max}} = 0.5$$

$$[(1-R)K_c - \Delta K] \frac{da}{dN} = C(\Delta K)^m \qquad K_c = 60 \text{ MPa}\sqrt{m}$$

$$\log\left\{ [(1-R)K_c - \Delta K] \frac{da}{dN} \right\} = \log C + m \log(\Delta K)$$

$$\overrightarrow{\log[(0.5 \times 60 - 5, 6) \times 5 \times 10^{-9}]} = \log C + m \log 5.6$$

$$\log[(0.5 \times 60 - 17.72) \times 10^{-6}] = \log C + m \log 17.72$$

$$\boxed{-6.914 = \log C + 0.748m}$$

$$-4.911 = \log C + 1.248m$$

 $C = 1.229 \ MN^{-3.3} \ m^{5.95}/\text{cycle}$, m = 4.006

C and m in Forman's model are different from these in Paris's model.

Dynamic fracture mechanics and rate effects 1. LEFM solution fields

9.2. Dynamics of moving crack tip, process zone size, crack speed

9.3. Crack path instabilities

Dynamic stress intensity factor

$$s^{ij}(r,\theta,t) = \frac{K_{I}(t)}{\sqrt{2\pi r}} \Sigma_{I}^{ij}(\theta(\hat{v}) + \frac{K_{II}(t)}{\sqrt{2\pi r}} \Sigma_{II}^{ij}(\theta(\hat{v})) \text{ as } r \to 0.$$

$$Crack speed$$

$$K_{I}(t) = \lim_{x_{2}\to 0} \sqrt{2\pi x_{2}} s^{11}(x_{2},0,t), \quad K_{II}(t) = \lim_{x_{2}\to 0} \sqrt{2\pi x_{2}} s^{12}(x_{2},0,t)$$

$$\Sigma_{I}^{11} = -\frac{1}{D} \left\{ (1 + \alpha_{II}^{2})^{2} \frac{\cos \frac{1}{2}\theta_{I}}{\sqrt{\gamma_{I}}} - 4\alpha_{I}\alpha_{II} \frac{\cos \frac{1}{2}\theta_{II}}{\sqrt{\gamma_{II}}} \right\},$$

$$\Sigma_{I}^{12} = \frac{2\alpha_{I}(1 + \alpha_{II}^{2})}{D} \left\{ \frac{\sin \frac{1}{2}\theta_{I}}{\sqrt{\gamma_{I}}} - \frac{\sin \frac{1}{2}\theta_{II}}{\sqrt{\gamma_{II}}} - 4\alpha_{I}\alpha_{II} \frac{\cos \frac{1}{2}\theta_{II}}{\sqrt{\gamma_{II}}} \right\},$$

$$\Sigma_{I}^{22} = \frac{1}{D} \left\{ (1 + \alpha_{II}^{2})(1 + 2\alpha_{I}^{2} - \alpha_{II}^{2}) \frac{\cos \frac{1}{2}\theta_{I}}{\sqrt{\gamma_{II}}} - 4\alpha_{I}\alpha_{II} \frac{\cos \frac{1}{2}\theta_{II}}{\sqrt{\gamma_{II}}} \right\},$$

$$\Sigma_{II}^{12} = \frac{2\alpha_{II}(1 + \alpha_{II}^{2})}{D} \left\{ \frac{\sin \frac{1}{2}\theta_{I}}{\sqrt{\gamma_{II}}} - \frac{\sin \frac{1}{2}\theta_{II}}{\sqrt{\gamma_{II}}} \right\},$$

$$\Sigma_{II}^{12} = \frac{1}{D} \left\{ 4\alpha_{I}\alpha_{II} \frac{\cos \frac{1}{2}\theta_{I}}{\sqrt{\gamma_{II}}} - (1 + \alpha_{II}^{2})^{2} \frac{\cos \frac{1}{2}\theta_{II}}{\sqrt{\gamma_{II}}} \right\},$$
Mode I
$$\Sigma_{II}^{22} = -\frac{2\alpha_{II}}{D} \left\{ (1 + 2\alpha_{I}^{2} - \alpha_{II}^{2}) \frac{\sin \frac{1}{2}\theta_{II}}{\sqrt{\gamma_{II}}} - (1 + \alpha_{II}^{2}) \frac{\sin \frac{1}{2}\theta_{II}}{\sqrt{\gamma_{II}}} \right\},$$

$$Mode II$$

Dynamic angular dependence function

$$\begin{split} \Sigma_I^{11} &= -\frac{1}{D} \left\{ (1 + \alpha_I^2)^2 \frac{\cos \frac{1}{2} \theta_I}{\sqrt{\gamma_I}} - 4\alpha_I \alpha_{II} \frac{\cos \frac{1}{2} \theta_{II}}{\sqrt{\gamma_{II}}} \right\} \\ \alpha_{(k)} &= \sqrt{1 - \hat{v}^2 / c_{(k)}^2} \\ \gamma_{(k)} &= \sqrt{1 - (\hat{v} \sin \theta / c_{(k)})^2} \end{split}$$

$$\tan \theta_{(k)} = \alpha_{(k)} \tan \theta, \quad k = 1, 2,$$
$$D = 4\alpha_I \alpha_{II} - (1 + \alpha_{II}^2)^2.$$

Longitudinal and shear wave speeds are

 $c_I := c_d$ and $c_{II} := c_s$

$$c_{\rm d} = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_{\rm s} = \sqrt{\frac{\mu}{\rho}}.$$

3D and 2D plane strain cd

$$a = \sqrt{\frac{E(1-v)}{\rho(1+v)(1-2v)}}, \quad c_s = \sqrt{\frac{E}{2\rho(1+v)}}, \quad c_R \approx c_s \frac{0.862 + 1.14v}{1+v}$$

Longitudinal (pressure) and shear waves



Dynamic fracture mechanics and rate effects LEFM solution fields

9.2. Dynamics of moving crack tip, process zone size, crack speed

9.3. Crack path instabilities

Rayleigh wave speed limit

$$\Sigma_I^{11} = -\frac{1}{D} \left\{ (1 + \alpha_I^2)^2 \frac{\cos\frac{1}{2}\theta_I}{\sqrt{\gamma_I}} - 4\alpha_I \alpha_{II} \frac{\cos\frac{1}{2}\theta_{II}}{\sqrt{\gamma_{II}}} \right\}$$

For one material under mode I maximum possible crack speed is Rayleigh wave speed where angular functions tend to infinity:

It can be shown that the *Rayleigh wave speed*, denoted by $c_{\rm R}$, equals the non-zero value of \hat{v} at which D vanishes (Rayleigh, 1885).



Dynamic stress intensity factor

$$K_{(k)}(t, a, \hat{v}) = k_{(k)}(\hat{v})K_{(k)}(t, a, 0)$$

stationary stress intensity factor, $K_{(k)}(t, a, 0)$, is the stress intensity factor that would result from the same applied loading if the crack tip were stationary at the instantaneous position corresponding to the crack length a

 $k_{(k)}(\hat{v})$ is a universal function of crack-tip speed for mode-(k) crack growth that is independent of the loading and the geometry of the body and that can be approximated as

$$k_{(k)}(\hat{v}) \approx (1 - \hat{v}/c_{\rm R})/\sqrt{1 - \hat{v}/c_{(k)}}$$

Note that k(v) approaches 0 as the crack speed tends to Rayleigh wave speed.

Dynamic energy release rate

• K, G relation:

$$G = \frac{1 - \nu}{2\mu} \left[A_{I}(\hat{v}) K_{I}^{2} + A_{II}(\hat{v}) K_{II}^{2} \right]$$

• Remember that for static case:
$$G = rac{1-
u}{2\mu} \left[|K_I^2 + K_{II}^2| \right]$$

The functions $A_{(k)}$ are universal functions that do not depend on the details of the loading or on the domain geometry. These functions have the properties,

• Static limit:
$$A_{(k)}
ightarrow 1 ext{ as } \hat{v}
ightarrow 0^+$$

• Rayleigh speed limit (G tends to infinity)

$$A_{(k)} = O[(c_{\rm R} - \hat{v})^{-1}]$$

Dynamic crack propagation criterion

• Dynamic Griffith criterion:

Crack propagates when energy release rate reaches

fracture toughness Λ_0 (resistance):

$$G = \Gamma_0$$

• Noting that:

$$K_{(k)}(t, a, \hat{v}) = k_{(k)}(\hat{v})K_{(k)}(t, a, 0),$$

$$G = \frac{1 - \nu}{2\mu} \left[A_I(\hat{v})K_I^2 + A_{II}(\hat{v})K_{II}^2 \right]$$

• For mode I we obtain:

$$\frac{1-\nu}{2\mu}A(\hat{v})k(\hat{v})^2 \left[K(t,a,0)\right]^2 = \Gamma_0$$

Dynamic crack propagation criterion

$$\frac{1-\nu}{2\mu}A(\hat{v})k(\hat{v})^2 \left[K(t,a,0)\right]^2 = \Gamma_0$$

$$\Leftrightarrow \quad g(\hat{v}) = \frac{2\mu\Gamma_0}{(1-\nu)\left[K(t,a,0)\right]^2}$$

 $g(\hat{v}) := A(\hat{v})k(\hat{v})^2$ is a universal function of the crack-tip speed that is very accurately approximated by

$$g(\hat{v}) \approx 1 - \hat{v}/c_{\rm R}$$
 for $0 \le \hat{v} \le c_{\rm F}$

Final note:



fracture toughness Λ_0 itself depends on crack speed v!

Example of dynamic fracture

- Infinite domain under dynamic mode I • load $\bar{\sigma} = 2\bar{t}_{\infty}$
- Static intensity factor is evaluated as, •

$$K(t, a, 0) = C\sqrt{2\pi c_{\rm d} t} \,\bar{\sigma}$$

Crack propagates when ullet

$$g(\hat{v}) = \frac{\mu \Gamma_0}{(1-\nu)\pi c_d \left(C\bar{\sigma}\right)^2 t}.$$

Crack initiation time then is, ullet

 au_0





Reminder:

Fracture process zone in dynamic fracture x_{T}

- Importance of process zone size A
 - Static estimate:

$$\begin{split} \Lambda &= \varsigma \pi \frac{\mu}{1-\nu} \frac{\phi}{\tilde{\sigma}^2} \propto \tilde{L} \\ \varsigma &= \begin{cases} \frac{1}{4} & \text{Dugdale model} \\ \frac{9}{16} & \text{Potential-based TSRs} \end{cases} \end{split}$$

- $\begin{array}{c} x_{\mathrm{T}} & x_{\mathrm{L}} \\ x_{\mathrm{CT}} & x_{\mathrm{CT}} \\ & & & & \\ & & & \\ & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\$
- Minimum number of elements in process zone size: There should be at least 4-10 elements along the PZ
- Dynamic estimate: PZS decreases as crack speed \hat{v} approaches Rayleigh wave speed c_R

$$\Lambda(\hat{v}) = \frac{\Lambda}{\Lambda(\hat{v})}, \quad \Lambda(\hat{v}) \to 0 \text{ as } \hat{v} \to c_{\mathrm{R}} \quad \Rightarrow$$

Smaller elements are needed in PZT as crack accelerates!



Small scale yielding in dynamic fracture



Super-shear crack propagation

Crack propagation speed background

- For a homogeneous solid, crack speed cannot exceed Rayleigh wave speed (c_R).
 - In practice, speed often does not exceed 50% of c_R.
 - Experiment with two weakly joined identical solids, where the weak interface confines the crack to the plane, an interfacial crack indeed approaches c_R.



Washabaugh, P. D. & Knauss, W. G. 1994 A reconciliation of dynamic crack velocity and Rayleigh wave speed in isotropic brittle solids. Int. J. Fracture 65, 97{114

Fig. 8. Dependence of the crack velocity on the property of the material (i.e. fractional strength or area fraction) preceding the crack front.

Do cracks do really reach Rayleigh speed?

Sharon Fineberg:

mirror, mist, hackle patterns as the crack accelerates





- Crack starts oscillating well before reaching Rayleigh wave speed V_R (c_R)
- Crack speed does not reach V_R (c_R)!
- For this material critical speed $v_c = 0.36 V_R$

Do cracks do really reach Rayleigh speed?

(2)

Confirming the continuum theory of dynamic brittle fracture for fast cracks



Beyond v_c , agreement breaks down owing to the appearance of the multiple-crack ensemble. But in this regime, the micro-branching process can momentarily produce a single-crack state which instantaneously attains its predicted single-crack velocity, for velocities up to $0.9c_R$.

- Crack dynamics does not have interia \rightarrow
- Crack reaches Rayleigh wave speed at instances

Cases that the crack speed exceed Rayleigh wave speed: Dissimilar material interfaces

Intersonic crack growth on an interface

By H. H. Yu[†] and Z. Suo

Department of Mechanical and Aerospace Engineering and Princeton Materials Institute, Princeton University, Princeton, NJ 08544, USA

- Crack can propagate on an interface between dissimilar solids at speeds between the smallest and the largest sonic speeds of the constituent solids.
 - Assuming the existence of a sharp crack tip:
 - The stress field is singular not only at the crack tip, but also along the shock front.
 - The singularity exponents differ from one half. \rightarrow
 - the energy release rate is either zero or infinite.

Crack propagation in homogeneous medium



Figure 5. Cohesive zone length as a function of the crack speed for weakly joined identical solids (opening mode).



Forbidden zone: $c_R < v < c_s$



Yu, H.H., Suo, Z., 2000b. Intersonic crack growth on an interface. Proceedings of the Royal Society of London, Series A (Mathematical, Physical and Engineering Sciences) 456, 223–46.

Possibilities of supershear crack propagation



Figure 7. Cohesive zone length as a function of the crack speed for an intersonic crack on the interface between an elastic and a rigid solid.



Yu, H.H., Suo, Z., 2000b. Intersonic crack growth on an interface. Proceedings of the Royal Society of London, Series A (Mathematical, Physical and Engineering Sciences) 456, 223–46.

Possibilities of supershear crack propagation



Fig. 5 Representative plot showing wave speeds for Homalite-100 and nomenclature of various regions

> Int J Fract (2007) 143:79–102 DOI 10.1007/s10704-007-9051-z

ORIGINAL PAPER

Simulation of dynamic crack growth using the generalized interpolation material point (GIMP) method

Nitin P. Daphalapurkar · Hongbing Lu · Demir Coker · Ranga Komanduri
Possibilities of supershear crack propagation: Burridge–Andrews mechanism



Fig. 7 (Online color) Snapshots of τ_{max} (MPa) for $V_{imp} = 30 \text{ m/s}$ at various durations (a) Stress wavefront arriving at the initial crack-tip (b) Stress wavefront loading the initial crack in predominantly shear mode (c) Crack has propagated along the interface after initiation at 10.4 μ s

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(d) Maximum shear stress pattern immediately after the transition mechanism (e) Formation of shear shock waves (f) Crack propagation at a sustained crack-tip velocity in intersonic regime ($c_R = 1164$ m/s)



Fig. 9 Plot of normalized (with c_R) crack-tip velocity with time for $V_{imp} = 30 \text{ m/s}$

- The sustained crack-tip velocity falls either in the sub-Rayleigh regime or in the region between $\sqrt{2}c_s$ and c_d
- MD simulation: Abraham FF, GaoH(2000) How fast can cracks propagate? Phys Rev Lett 84(40):3113–3116
- Shear dominated crack initially accelerates to c_R
- Followed by the nucleation of a daughter microcrack ahead of the main crack
- Finally coalescence of the mother and daughter-
- cracks with the crack-tip velocity reaching a
- value as high as the longitudinal wave speed.
- <u>After coalescence</u>, when the far-field loading is relaxed,
- the crack decelerates and propagates at a steady
- rate close to a speed of $\sqrt{2}c_s$.

Possibilities of supershear crack propagation: MACH cones



Near-source ground motion from steady state dynamic rupture pulses

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GEOPHYSICAL RESEARCH LETTERS, VOL. 32, L03302, doi:10.1029/2004GL021793, 2005

Possibilities of supershear crack propagation

Earthquake and wave propagation



Collaborators: R. Haber, J. Erickson, A. Elbanna (UIUC)

TPV205-2D with Reduced d_c Supershear Rupture: Mach Cones





9. Dynamic fracture mechanics and rate effects

9.1. LEFM solution fields

9.2. Dynamics of moving crack tip, process zone size, crack speed

9.3. Crack path instabilities

Do cracks accelerate to Rayleigh speed?

Sharon Fineberg:

mirror, mist, hackle patterns as the crack accelerates





- Crack starts oscillating well before reaching Rayleigh wave speed V_R (c_R)
- Crack speed does not reach V_R (c_R)!
- For this material critical speed $v_c = 0.36 V_R$

Sample fracture patterns for brittle materials?

Branching only occurred for thicker specimens and very short notch depths



Fig. 5. Macroscopically planar fracture surfaces for the specimens containing the (a) 2.0 mm, (b) 1.4 mm, (c) 1.0 mm and (d) 0.5 mm notches.

Successful branching appears to take place when micro-cracks form a sufficient distance from the main crack tip to allow their subsequent propagation to take place as opposed to being arrested by the arrival of an unloading wave from the main crack.

Murphy 2006, Dynamic crack bifurcation in PMMA

Sample fracture patterns for brittle materials

1. Too much energy: a single crack cannot dissipate that much energy particularly in brittle materials



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Effect of T-Stress

Murphy 2006, Dynamic crack bifurcation in PMMA

Further developing this theme, Streit and Finnie [6] suggested that the directional stability of a quasi-static crack should be influenced by the state of stress at a finite distance ahead of the crack tip in the location where the micro-cracks or voids are formed. In this case, not only the singular terms in the stress expansions, but also the remote normal stress in the crack propagation direction should be considered. The latter is known as the *T*-stress, and its influence on crack stability is well known [7]. This concept was applied to dynamic crack curving by Ramulu and Kobayashi [8]. In accordance with previous findings, Ramulu et al. [9] observed that successful branching always took place at a constant value of $K_{\rm I}$, which was therefore cited as a necessary

[6] Streit R, Finnie I. An experimental investigation of crack-path directional stability. Exp Mech 1980;20:17–23.
[7] Cotterell B, Rice JR. Slightly curved or kinked cracks. Int J Fracture 1980;16:155–69.

[8] Ramulu M, Kobayashi AS. Dynamic crack curving—a photoelastic evaluation. Exp Mech 1983;23:1–9.
[9] Ramulu M, Kobayashi AS, Kang BSJ, Barker DB. Further studies on dynamic crack branching. Exp Mech 1983;23:431–7.



Figure 11. Crack paths observed by Radon et al. [29] in experiments on biaxially stressed PMMA sheets.

- A critical role of the sign of the T -stress applies only to the situation of single crack growing in a large plate
- Counter examples: Array of collinear cracks, wedge cracks, etc.

Melin, Solveig. "The influence of the T-stress on the directional stability of cracks." International Journal of Fracture 114.3 (2002): 259-265. 443

Crack branching K-critical value Relation to T-stress

Ramulu M, Kobayashi AS, Kang BSJ, Barker DB. Further studies on dynamic crack branching. Exp Mech 1983;23:431–7

Branching takes place at a constant value of KI

) A crack-branching stress-intensity factor of $K_{lb} = 3.3 \text{ MPa}\sqrt{m}$ and a characteristic radius of $r_c = 0.75 \text{ mm}$ were determined for this polycarbonate sheet.

T < 0 has suppressing effect

Crack curving of post-branched cracks, attraction and repulsion, depend not only on K_{II}/K_I but more importantly on σ_{ox} . Negative σ_{ox} suppresses crack curving irrespective of the sign of K_{II}/K_I .



Energetic argument for crack branching

Eshelby JD (1970) Energy relations and the energy–momentum tensor in continuum mechanics. In: Kanninen MF, Adler WF, Rosenfield AR, Jaffee RI (eds) Inelastic behaviour of solids. McGraw-Hill, New York, pp 77–115

Branching takes place when the energy for 1 crack is equal to the energy of 2 bifurcated cracks

In the single crack case, a growth criterion for a branched crack must be based on the equality between the energy flux into the two propagating tips and the energy required to open the material and create new surfaces as a result of this propagation (Eshelby 1970).

Adda-Bedia M (2005) Brittle fracture dynamics with arbitrary paths- III. The branching instability under general loading. J Mech Phys Solids 53:227–248

- The jump in the energy release rate due to branching is maximized when the **branches start to propagate very slowly**.
- The branching of a single propagating crack under tensile loading was found to be energetically possible when its speed exceeds a certain critical value (Adda-Bedia 2005).



Fig. 12 Schematic representation of a straight crack with symmetrically branched curved extensions. The fictitious straight crack around which the perturbation expansion is performed is drawn on the lower branch

Katzav, E., M. Adda-Bedia, and Rodrigo Arias. "Theory of dynamic crack branching in brittle materials." International Journal of Fracture 143.3 (2007): 245-271.