1. (**50 Points**). For each of the scenarios below use the letter of B or D for transition to be more brittle or ductile, respectively. In each case, use **less than two sentences** to explain your choice.
   (a) Thicker plates.
   (b) Temperature rise for BCC metals.
   (c) Radiation for BCC metals.
   (d) Smaller grain size.
   (e) High strain rate loading.
   (f) Larger domain size.

   Mention one example from the list above that ductility and strength decrease/increase together.

2. (**50 Points**). Briefly answer the following questions.
   (a) What is the relation between K_I and G for pure mode I and plane stress condition?
   (b) Which of these quantities: plastic radius, singular radius, or CTOD, scale as $\left(\frac{K}{\sigma_y}\right)^2$?
   (c) Between plane stress and plane strain which one results in a larger plastic zone? In **less than 2 sentences** explain your choice.

3. (**100 Points**). A centered crack large plate (*i.e.,* you can use mid-crack expression for an infinite domain) is subjected to a uniform tension $\sigma = 300$ MPa perpendicular to the crack plane. Yield stress $\sigma_y = 860$ MPa, $K_{Ic} = 100\ \mathrm{MPa}\sqrt{\mathrm{m}}$, and Poisson ratio $\nu = 0.2$. Assuming plane strain mode,
   (a) Using a purely LEFM analysis calculate the maximum crack length $2a_{\mathrm{LEFM}}$ in mm unit the plate can withstand without failure.
   (b) Instead of a full PFM analysis we can adjust *crack length* in SIF expression $K = \sqrt{\pi a_{\mathrm{eff}}}\sigma$ to take plasticity at the crack tip into account. Compute a_{PFM} in mm unit the plate can withstand without failure by adjusting crack length $a_{\mathrm{eff}} = a + r_p$, $r_p = \frac{1}{2\pi}\left(\frac{K_{I_{\mathrm{eff}}}}{\sigma_{ys}}\right)^2$ (make sure to use the right equation for σ_{ys} in terms of σ_y for plain strain condition). Compare $2a_{\mathrm{LFEM}}$ and $2a_{\mathrm{PFM}}$ and explain their difference.
   (c) Find r_p corresponding to a_{PFM} in mm.

4. (**100 Points**). Assume that for the last problem the plate thickness is in fact is $B = 10$ mm.
   (a) According to ASTM E 399 determine if the plate is plane strain or plane stress mode.
   (b) Calculate critical stress intensity factor K_c for the given thickness.
   (c) Based on the value of K_c calculate the maximum crack length $2a_{\mathrm{LFEM-th}}$ in mm unit the plate can withstand without failure. You do not need to adjust crack length $a_{\mathrm{eff}} = a + r_p$.
   (d) Compare $2a_{\mathrm{LFEM-th}}$ and $2a_{\mathrm{LFEM}}$ and explain their difference.

5. (**50 Points**). Based on numerical results we have obtained $J := J_1 = 50.5\ \mathrm{GPa-m}$ and $J_2 = -10\ \mathrm{GPa-m}$ (J integrals are energy release rates normalized by unit crack *area* advance here). Elastic modulus is 200 GPa What are possible solutions for K_I and K_{II}? Use $\nu = 0$ in your computation. How do you choose the right pair of solutions K_I and K_{II} in practice?

6. (**100 Points**). A large thick plate contains a crack of length $2a_0$ and is subjected to a constant amplitude tensile cyclic stress normal to the crack with maximum stress $\sigma_{\max} = 200$ MPa and stress range $\Delta\sigma = \sigma_{\max} - \sigma_{\min}$, $\sigma_{\min} = 0$. The fatigue crack growth is governed by equation,

$$\frac{\mathrm{d}a}{\mathrm{d}N} = 3.9 \times 10^{-14}(\Delta K)^{3.7} \tag{1}$$

   where $\frac{\mathrm{d}a}{\mathrm{d}N}$ is expressed in m/cycle and ΔK in $\mathrm{MPa}\sqrt{\mathrm{m}}$.

(a) Determine a_f. Use $K_{Ic} = 24$ MPa$\sqrt{\text{m}}$, and use units $\Delta\sigma$ MPa, K : MPa$\sqrt{m}$, length: m in all your calculations.

(b) Determine the fatigue lifetime of the plate for $2a_0 = 2$ mm,

7. **OPTIONAL:** (**200 Points**) Probabilistic analysis of fatigue:

Consider a plate with one initial crack length a_0 where a_0 follows a *probability density function* (PDF) $p(a_0)$. The *cumulative density function* (CDF) is defined as $P(z) = \mathcal{P}(a_0 \leq z)$, $where \mathcal{P}(a_0 \leq z)$ is the probability of a_0 being smaller than a given length z. CDF is the main function that characterizes the distribution of a random variable and when it is differntiable we have, $p(z) = \frac{\mathrm{d}P(z)}{\mathrm{d}z}$. For this example, we consider a simplified, yet nonphysical distribution of initial crack lengths. We assume a_0 follow a uniform distribution of crack lengths between $a_{\min}$ and $a_{\max}$: $U(a_{\min}, a_{\max})$. That is,

$$p(z) = \begin{cases} \frac{1}{a_{\max} - a_{\min}} & a_{\min} \leq z \leq a_{\max} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

For questions below use parameters from problem 6 with the addition $a_{\min} = 0.1$ mm, $a_{\max} = 3$ mm (Do not forget to change sizes to m before any calculation).

(a) Find CDF $P(z)$.

(b) Median corresponds to a value where 50% of values are smaller and 50% are larger than that value. That is, $P(a_{0.50}) = P(a_{\text{median}}) = 0.5$. Find, $a_{\text{median}} = a_{0.50}$.

(c) Mean value of a function $g(z)$ of PDF $p(z)$ (denoted by $\mathbb{E}[g]$) is defined by,

$$\mathbb{E}[g] = \int_{-\infty}^{\infty} g(z)p(z)\mathrm{d}z \tag{3}$$

so the mean value if initial crack length is ($g(a_0) = a_0$) $\quad \Rightarrow g =$ identity map)

$$a_{\text{mean}} = \mathbb{E}[a_0] = \int_{-\infty}^{\infty} z p(z)\mathrm{d}z \tag{4}$$

Find a_{mean} and compare it with a_{median}.

(d) Find $a_{0.95}$ the initial crack length where 95% of initial cracks are shorter than that (Use $P(a_{0.95}) = 0.95$).

(e) Now, we want to characterize the life of the specimen given the PDF of initial cracks. Assume that the crack is inside a very large domain, perpendicular to the applied stress. Then $K_I = \sigma\sqrt{\pi a}$ for a crack length a ($Y = 1$ on slide 373). From slide 373 we can write,

$$N_f(a_0) = \gamma \left[\frac{1}{a_0^{(m-2)/2}} - \frac{1}{a_f^{(m-2)/2}} \right], \quad \text{where} \quad \gamma = \frac{2}{(m-2)CY^m(\Delta\sigma)^m \pi^{m/2}} \tag{5}$$

for $m > 2$. The values of m, C, $\Delta\sigma$, a_f are the same as problem 6 and $Y = 1$.
Find γ.

(f) Find the mean number of cycles.

Hint: Note $N_f(z) = \gamma \left[\frac{1}{z^{(m-2)/2}} - \frac{1}{a_f^{(m-2)/2}} \right]$ and use (2) and (4) to evaluate ${N_f}_{\text{mean}} = \mathbb{E}[N_f]$ (notation for mean value).

(g) To determine the number of cycles $(N_f)_\alpha$ where with probability α the plate will withstand the loads (and a does not reach catastrophic crack length a_f) we note that,

$$\mathcal{P}(N_f \geq (N_f)_\alpha) = \alpha \quad \Rightarrow$$

$$\mathcal{P}(a \leq a_\alpha) = \alpha, \text{ where } (N_f)_\alpha = N_f(a_\alpha) = \gamma \left[\frac{1}{a_\alpha^{(m-2)/2}} - \frac{1}{a_f^{(m-2)/2}} \right], \quad \text{but}$$

$$\mathcal{P}(a \leq a_\alpha) = P(a_\alpha) = \alpha \quad \Rightarrow a_\alpha = P^{-1}(\alpha)$$

So, to obtain $(N_f)_\alpha$ we use,

$$\boxed{a_\alpha = P^{-1}(\alpha), \quad \Rightarrow \quad (N_f)_\alpha = N_f(a_\alpha) = \gamma \left[\frac{1}{a_\alpha^{(m-2)/2}} - \frac{1}{a_f^{(m-2)/2}} \right]} \tag{6}$$

to compute $(N_f)_\alpha$.

Compute $(N_f)_{0.50}$ which is the *median* of number of cycles. That is, this value corresponds to a life cycle where half of the cases the plate have a shorter life and half a longer life. Note that, we have already computed $a_{0.50}$ in 7(b). You just need to plug that into (6).

(h) In designs we want to do inspections when the probability of failure is a very small number, that is probability of safety is very high. For this example, we consider the threshold to be 95% (which can be very unsafe). If we want to inspect the plate such that it is safe with 95% confident, what would be the number of cycles between current state and the next inspection? **Hint**: In (6) you need to plug in $a_{0.95}$ which is already calculated in 7(d).

(i) We already have computed ${N_f}_{\text{mean}} = \mathbb{E}[N_f]$ from 7(f). This is the expected value of number of cycles that the plate will not fail. We can also compute N_f based on the mean value of crack length. The second approach is what we do in deterministic analyses. That is, based on the mean value of input values, compute the output of the system and hope that to be a good representative of the mean value of the response. Herein, we want to compare the two. The response of the mean value is simply ${N_f}_{0.50} = N_f(a_{0.50})$ (because from 7(c) $a_{\text{median}} = a_{0.50} = a_{\text{mean}}$).

Discuss how the actual mean value of life cycles (${N_f}_{\text{mean}} = \mathbb{E}[N_f]$ from 7(f)) compares with the response of the mean value of crack lengths (${N_f}_{0.50}$ from 7(h)).